

# THE $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$ COSET THEORY AS A HAMILTONIAN REDUCTION OF $\widehat{D}(2|1; \alpha)$

B. L. FEIGIN AND A. M. SEMIKHATOV

**ABSTRACT.** We show that the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}/\widehat{sl}(2)_{k_1+k_2}$  is a quantum Hamiltonian reduction of the exceptional affine Lie superalgebra  $\widehat{D}(2|1; \alpha)$  and that the corresponding  $W$  algebra is the commutant of the  $\mathcal{U}_q D(2|1; \alpha)$  quantum group.

## CONTENTS

1. Introduction	1	4.2. The $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$ coset	19
2. Fermionic screenings for $WD(2 1; \alpha)$	7	4.3. Reconstructing $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$ currents	21
2.1. A cubic relation on three fermionic screenings	7	4.4. Reconstructing $\widehat{sl}(2)_{k_i}$ vertex operators	22
2.2. The commutant of the screenings	9	5. Hamiltonian reduction of $\widehat{D}(2 1; \alpha)$	23
2.3. Bosonic screenings	11	5.1. Remarks on Hamiltonian reduction	23
3. $WD(2 1; \alpha)$ by deformation and the $WD(2 1; \alpha)$ vertex operators	12	5.2. Hamiltonian reduction of $\widehat{D}(2 1; \alpha)$	24
3.1. A “ $\beta$ -deformation” of $\widehat{sl}(2)$	12	6. Discussion and conclusions	27
3.2. 122-type quantum group representations	14	6.1. $WD(2 1; \alpha)$ from BFB screenings	27
3.3. Symmetric bosonization and the triplet operator	15	6.2. Matter realization of $WD(2 1; \alpha)$	28
4. The vertex operator reconstruction	17	6.3. Quantum group representations	30
4.1. Quantum groups and vertex-operator extensions	17	6.4. Resolutions and characters	30
		Appendix A. The dimension-four field	33
		Appendix B. The $\widehat{D}(2 1; \alpha)$ algebra	35
		References	37

## 1. INTRODUCTION

Extensions of bosonic algebras via vertex operators have been seen to demonstrate a remarkable appearance of affine Lie *superalgebras*: two  $\widehat{sl}(2)$  algebras with the levels  $k$  and  $k'$  related by  $(k+1)(k'+1) = 1$  are extended via their spin- $\frac{1}{2}$  vertex operators to the affine superalgebra  $\widehat{sl}(2|1)$  [1]. Moreover, a class of representations of the exceptional affine Lie superalgebra  $\widehat{D}(2|1; \alpha)$  [2, 3] can also be realized by extending  $\widehat{sl}(2)_k \oplus \widehat{sl}(2)_{k'}$  by spin- $\frac{1}{2}$  vertex operators [1]. The emergence of the  $\widehat{D}(2|1; \alpha)$  algebra is rather intriguing, and this suggests looking for other occurrences of  $\widehat{D}(2|1; \alpha)$  or related algebras hidden behind some known conformal field theory structures.

**1.1. Formulation of the main result.** We show that  $\widehat{D}(2|1; \alpha)$  is related to a well-known object, the coset conformal theories  $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$ , via the (quantum) Hamiltonian reduction,

$$(1.1) \quad \text{Quantum Hamiltonian Reduction}(\widehat{D}(2|1; \alpha)_{\varkappa}) = \frac{\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}}{\widehat{sl}(2)_{k_1+k_2}},$$

with the  $\alpha$  parameter of the algebra and its level  $\varkappa$  expressed as

$$(1.2) \quad \alpha = -1 - \frac{k_1 + 2}{k_2 + 2}, \quad \varkappa = \frac{-1}{k_1 + k_2 + 4}.$$

We also show that

$$(1.3) \quad \frac{\widehat{s\ell}(2)_{k_1} \oplus \widehat{s\ell}(2)_{k_2}}{\widehat{s\ell}(2)_{k_1+k_2}} = \mathcal{WD}(2|1; \alpha),$$

where  $\mathcal{WD}(2|1; \alpha)$  is the  $W$  algebra determined by the root system of the Lie superalgebra  $D(2|1; \alpha)$ ; the coset theory on the left-hand side can be defined as the BRST cohomology of the complex associated with the  $\widehat{s\ell}(2)_{-4}$  algebra diagonally embedded in  $\widehat{s\ell}(2)_{k_1} \oplus \widehat{s\ell}(2)_{k_2} \oplus \widehat{s\ell}(2)_{k_3}$ , where  $k_1 + k_2 + k_3 = -4$ . The central charge is therefore given by

$$(1.4) \quad c = \frac{3k_1}{k_1 + 2} + \frac{3k_2}{k_2 + 2} - \frac{3(k_1 + k_2)}{k_1 + k_2 + 2}.$$

Two general remarks are in order. First, the algebra  $\widehat{D}(2|1; \alpha)$  admits *different* Hamiltonian reductions, depending on the chosen maximal nilpotent subalgebra. Relations (1.2) apply to the case where the three simple roots are chosen fermionic.<sup>1</sup> Second, the mapping between  $(k_1, k_2)$  and  $(k, \alpha)$  is not uniquely fixed because on the one hand, the  $\alpha$  parameter is defined modulo an order-6 group of discrete transformations and on the other hand, the BRST construction of the coset is invariant under transpositions of the three levels  $k_1$ ,  $k_2$ , and  $k_3 = -k_1 - k_2 - 4$ ; this also applies to writing the coset in the GKO form [7], for example, the left-hand side of (1.3) can be replaced with  $\widehat{s\ell}(2)_{k_2} \oplus \widehat{s\ell}(2)_{-k_1-k_2-4} / \widehat{s\ell}(2)_{-k_1-4}$ .

Although the notation  $\mathcal{WD}(2|1; \alpha)$  for the  $W$  algebra explicitly indicates the root system that determines this algebra, it does not specify the central charge; we sometimes use the notation  $\mathcal{WD}_{2|1}(k_1, k_2)$  for this  $W$  algebra with the central charge in Eq. (1.4). Similarly, the level  $\varkappa$  in the notation  $\widehat{D}(2|1; \alpha)_{\varkappa}$  is conventionally taken to be the level of one of the  $\widehat{s\ell}(2)$  subalgebras in  $\widehat{D}(2|1; \alpha)$ ; a more convenient way to fix both the  $\alpha$  parameter and the level is to specify the levels  $\varkappa_1$ ,  $\varkappa_2$ , and  $\varkappa_3$  of three  $\widehat{s\ell}(2)$  subalgebras of  $\widehat{D}(2|1; \alpha)$  (these levels are related by  $\frac{1}{\varkappa_1} + \frac{1}{\varkappa_2} + \frac{1}{\varkappa_3} = 0$ ); in terms of  $k_1$  and  $k_2$ , we then have

$$(1.5) \quad \varkappa_1 = \frac{1}{k_1 + 2}, \quad \varkappa_2 = \frac{1}{k_2 + 2}, \quad \varkappa_3 = \frac{-1}{k_1 + k_2 + 4}.$$

The strategy to arrive at the results in Eqs. (1.1)–(1.3) involves a combination of several methods, which are outlined in what follows.

**1.2. Quantum groups and Hamiltonian reduction.** Quantum groups play an important role in the theory of vertex operator algebras, similar to the role of a symmetry group. This has various manifestations, one of these being the Kazhdan–Lusztig correspondence [8] between representation categories of a vertex operator algebra and of the corresponding quantum group (in particular cases, equivalence of quasitensor categories has been proved). Another important collection of examples is

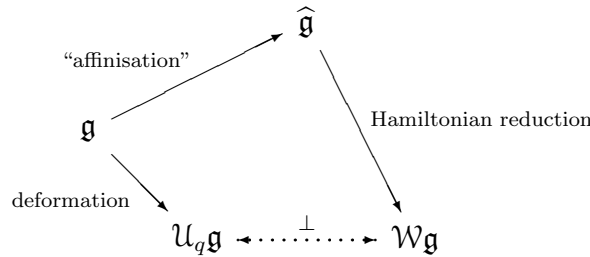
---

<sup>1</sup>We only consider Hamiltonian reductions that are “*maximal*” in that all the nilpotent subalgebra currents are constrained. A partial Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$  leads to (nonlinear)  $N=4$  superconformal algebras [4, 5], which raises an interesting question regarding a “secondary” Hamiltonian reduction [6] from the  $N=4$  superconformal algebra to the coset.

provided by *free-field realizations*, where the role of the appropriate quantum group is reminiscent of the Galois group: the quantum group can be thought of as acting by symmetries on the free-field space such that the invariants of this action make up a vertex operator algebra. In practice, the nilpotent (upper-triangular) quantum group generators are the *screening operators*; in other words, the vertex operator algebra generators are singled out by the condition that they commute with the screenings. This vertex operator algebra is said to be the *commutant* of the quantum group in the free-field space (the commutant is always understood as a subalgebra in the algebra of vacuum descendants of the free-field theory).

Vertex operator algebras defined as the commutant of a chosen set of screening operators are  $W$  algebras. The general pattern emerging from a number of known examples is that interesting  $W$  algebras typically arise when the screenings satisfy some special relations. For generic operators taken as screenings, the corresponding commutant is trivial; the condition for Virasoro generators to exist in the commutant restricts the screenings, and the existence of a larger algebra requires more relations to be satisfied by the screenings, which makes the corresponding quantum group “smaller.”<sup>2</sup>

A popular class of  $W$  algebras are associated with root systems of semisimple finite-dimensional Lie algebras. The screening operators  $\sigma_i = \oint e^{\vec{a}_i \cdot \vec{\varphi}}$  are then constructed from a set of free fields  $\vec{\varphi}$  by taking simple root vectors  $\vec{\alpha}_i$  and rescaling them into  $\vec{a}_i = \frac{1}{\kappa} \vec{\alpha}_i$  (with a parameter related to the central charge of the  $W$  algebra resulting from the reduction). These  $W$  algebras can also be obtained via *Hamiltonian reduction*.<sup>3</sup> Given a system of screening operators corresponding to the (upper-triangular) Chevalley generators of a quantum group and then taking the corresponding Lie algebra (assumed to be finite-dimensional and semisimple) and constructing its affinisation, one expects to arrive at the algebra whose Hamiltonian reduction is the commutant of the screenings. This can be schematically represented as



where ‘ $\perp$ ’ indicates that each of the two objects is the commutant of the other in the appropriate free-field space. Conversely, the Hamiltonian reduction of an affine Lie algebra  $\widehat{\mathfrak{g}}$  (with a semisimple finite-dimensional  $\mathfrak{g}$ ) is generally expected to be the commutant of the quantum group  $\mathcal{U}_q \mathfrak{g}$ .

<sup>2</sup>A precise criterion for the occurrence of a nontrivial  $W$  algebra is not known, however; for example, it is *not* true that the corresponding quantum group must necessarily be the  $q$ -deformation of a finite-dimensional semisimple Lie algebra.

<sup>3</sup>by which we mean the quantum Drinfeld–Sokolov reduction—the *maximum* Hamiltonian reduction “with characters,” i.e., such that the reduction constraints imposed on simple root operators are given by  $e_i(z) - \chi(e_i(z)) = 0$  with a character  $\chi : \widehat{\mathfrak{n}} \rightarrow \mathbb{C}$ .

Formulating the general scheme outlined in the above diagram as a theorem would face several subtle points. The quantum group is only sensitive to the *exponentials* of the scalar products of the momenta  $\vec{a}_i$  used in constructing the screening operators, but the commutant of the screenings can depend on the actual scalar products; this raises the problem of finding a “preferred” matrix of scalar products. For bosonic algebras (i.e., in the cases where  $\mathfrak{g}$  is *not* a Lie superalgebra), the recipe that is known to work amounts to taking the scalar products of the momenta to be precisely (up to a common factor) the scalar products of the corresponding root vectors. This is not necessarily so for superalgebras, and the general reformulation of the above scheme is not known in that case. In fact, even the definition of the  $W$  algebra  $\mathcal{W}\mathfrak{g}$  determined by the root system of  $\mathfrak{g}$  requires a clarification in the case where  $\mathfrak{g}$  is a Lie superalgebra. The proposal given in what follows applies to the case where the odd roots are isotropic.

**1.3. Odd roots, fermionic screenings, and  $\mathcal{U}_q D(2|1; \alpha)$ .** For a Lie superalgebra  $\mathfrak{g}$  all of whose odd roots are isotropic, we extend the notation  $\mathcal{W}\mathfrak{g}$  to denote the algebra that is defined as in the bosonic case but with the Chevalley operators corresponding to each odd isotropic root  $\alpha_i$  (i.e.,  $\vec{\alpha}_i \cdot \vec{\alpha}_i = 0$ ) replaced with  $\oint e^{\vec{a}_i \cdot \vec{\varphi}}$  where  $\vec{a}_i$  are determined by  $\vec{a}_i \cdot \vec{a}_i = 1$  and by the condition that the set of all screenings satisfy the nilpotent subalgebra of  $\mathcal{U}_q \mathfrak{g}$ .

Operators  $\oint e^{\vec{a}_i \cdot \vec{\varphi}}$  with  $\vec{a}_i \cdot \vec{a}_i = 1$  are called *fermionic screenings*; all other screenings are indiscriminately called *bosonic*.

Examples of systems with fermionic screenings are provided by two-boson realizations, where two fermionic screenings determine the nilpotent subalgebra of  $\mathcal{U}_q s\ell(2|1)$ ; its commutant is a nontrivial  $W$  algebra, which also is the Hamiltonian reduction of  $\widehat{s\ell}(2|1)$ . (This justifies the notation  $\mathcal{W}s\ell(2|1)$  for this  $W$  algebra, even though the condition  $\vec{a} \cdot \vec{a} = 1$  is not read off from the Cartan matrix of  $s\ell(2|1)$ ; this  $W$  algebra is also isomorphic to  $\widehat{s\ell}(2)/u(1)$ ). For a system of one fermionic and one bosonic screening, the resulting  $W$  algebra is isomorphic to  $\mathcal{W}s\ell(2|1)$  constructed using two fermionic screenings.

In this paper, we consider the case with three fermionic screenings  $(\sigma_1, \sigma_2, \sigma_3)$ . With a single relation imposed on these operators in the grade  $(1, 1, 1)$  with respect to  $(\sigma_1, \sigma_2, \sigma_3)$ , *the three fermionic screenings generate the nilpotent subalgebra of the quantum group  $\mathcal{U}_q D(2|1; \alpha)$* . This is the origin of  $D(2|1; \alpha)$ -related algebras in this paper.

The  $\mathcal{U}_q D(2|1; \alpha)$ -relation imposed on the fermionic screenings translates into a relation on the scalar products of the momenta of the screenings,

$$\vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{a}_3 + \vec{a}_2 \cdot \vec{a}_3 = n \in \mathbb{Z},$$

which involves an arbitrary integer. We analyze the commutant of  $(\sigma_1, \sigma_2, \sigma_3)$  for the existence of higher-dimension operators, in addition to the Virasoro generators. The first of these operators can occur at dimension 4. We find that for generic values of the other parameters, a primary

dimension-4 field exists in the commutant if and only if  $n = -1$ . We conjecture that for generic values of the other parameters, a nontrivial  $W$  algebra exists in the commutant of  $(\sigma_1, \sigma_2, \sigma_3)$  if and only if  $n = -1$  (the *conjecture* consists in the “only if” part, the converse is a part of what is proved below). We define the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$  to be the commutant of the three fermionic screenings that represent the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  with  $n = -1$  in the above equation for the scalar products in the three-boson space.

To mention another example of the occurrence of a fermionic screening in the three-boson case, we note that the system of one fermionic and two bosonic screenings, with the bosonic screenings commuting with each other, gives the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  corresponding to a different choice of simple roots. The commutant is the same  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$  as determined by the three fermionic screenings.

The  $\mathcal{WD}(2|1; \alpha)$  algebra is similar to other known  $W$  algebras in that it allows *two* systems of screening operators generating different quantum groups that are Langlands-dual to each other (and are therefore different in the non-simply-laced case). For  $\mathcal{WD}(2|1; \alpha)$ , these quantum groups are  $\mathcal{U}_q D(2|1; \alpha)$  and  $\mathcal{U}_{q_1} sl(2) \otimes \mathcal{U}_{q_2} sl(2) \otimes \mathcal{U}_{q_3} sl(2)$ . The latter is represented by three commuting bosonic screenings that play an important role in establishing the relation of  $\mathcal{WD}(2|1; \alpha)$  to the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}/\widehat{sl}(2)_{k_1+k_2}$ .

**1.4. Vertex-operator extensions of vertex operator algebras.** To show that  $\mathcal{WD}(2|1; \alpha)$  is the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}/\widehat{sl}(2)_{k_1+k_2}$ , we reconstruct the two  $\widehat{sl}(2)_{k_i}$  algebras in the numerator from the  $\widehat{sl}(2)_{k_1+k_2}$  algebra in the denominator and the  $\mathcal{WD}(2|1; \alpha)$  algebra. This is done by constructing the *vertex-operator extension*

$$(1.6) \quad \mathcal{WD}(2|1; \alpha) \otimes \mathcal{U}\widehat{sl}(2)_{k_1+k_2} \longrightarrow \mathcal{U}\widehat{sl}(2)_{k_1} \otimes \mathcal{U}\widehat{sl}(2)_{k_2}$$

involving vertex operators that carry representations of the three  $\mathcal{U}_q sl(2)$  quantum groups.

Vertex-operator extensions highlight the use of quantum groups in describing *monodromy properties* of vertex operators; extensions with the help of vertex operators as, e.g., in (1.6), require local operators, and these are usually constructed by taking products of operators carrying dual quantum group representations and then taking the “quantum” trace so as to obtain operators with trivial monodromies with respect to each other.

A given set of screening operators can be used to define, in addition to the commutant  $W$  algebra, a natural set of vertex operators of this  $W$  algebra (and their descendants) by selecting all those free-field operators that generate *finite-dimensional* quantum group representations under the action of the screenings. These vertex operators are therefore labeled by representations of the quantum group(s) generated by the screenings. Vectors in the vacuum representation of the  $W$  algebra—or equivalently, the local fields—are then in a 1 : 1 correspondence with quantum-group singlets.

We use vertex-operator extensions to construct the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$  currents and also the cor-

responding vertex operators (in all cases, we only use the  $\mathcal{WD}(2|1; \alpha)$  vertex operators that are singlets with respect to  $\mathcal{U}_q D(2|1; \alpha)$ ). Constructing the  $\widehat{sl}(2)_{k_i}$  vertex operators is based on the “contraction” given by the quantum trace in the product of the two-dimensional  $sl(2)$  quantum group representations  $\mathbb{C}_q^2$  and  $\mathbb{C}_{q'}^2$  with the “dual” quantum group parameters  $q = e^{2\pi i/(k+2)}$  and  $q' = e^{2\pi i/(k'+2)}$ . The idea of this “duality,” with  $\frac{1}{k+2} + \frac{1}{k'+2} \in \mathbb{N}$ , was discussed in [1] (which was devoted to the case where  $\frac{1}{k+2} + \frac{1}{k'+2} = 1$ ) and is similar to the duality used in matter + gravity theory and also has a counterpart in solvable lattice models of statistical mechanics, where expressing the  $\mathbf{T}$  operators as the trace of  $\mathbf{L}$  operators is parallel to the above contraction of vertex operators. In the present case, we have  $\frac{1}{k+2} + \frac{1}{k'+2} = 0$ , which allows us to take the quantum trace of the product of  $\widehat{sl}(2)_{k_1+k_2}$  and  $\mathcal{WD}(2|1; \alpha)$  vertex operators. This gives the spin- $\frac{1}{2}$  vertex operators for each of the  $\widehat{sl}(2)_{k_i}$  algebras (the quantum groups that are not involved in the contraction become quantum group symmetries of the resulting vertex operators); the spin- $\frac{1}{2}$  operators generate the entire algebra of vertex operators. The  $\widehat{sl}(2)_{k_i}$  currents can be reconstructed via a similar vertex-operator extension involving a contraction in the product of three-dimensional quantum group representations (such that the result is a singlet with respect to all quantum groups).

**1.5. Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$ .** We perform the Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$  with the help of a “BRST” operator  $\mathcal{Q}$  implementing the constraints imposed on the nilpotent subalgebra currents. The reduction is such that the cohomology of  $\mathcal{Q}$  certainly contains a Heisenberg algebra  $\mathcal{H}_0$ , which is a “trivial” piece guaranteed by the nature of the reduction, while the nontrivial problem is to find a  $W$  algebra commuting with  $\mathcal{H}_0$ . The Virasoro generators of this  $W$  algebra can be constructed explicitly, and it is then verified that their central charge is the one in Eq. (1.4). To show that the cohomology contains the entire  $\mathcal{WD}(2|1; \alpha)$  algebra, we introduce a filtration on the BRST complex such that the BRST operator splits as  $\mathcal{Q} = \mathcal{Q}^{(0)} + \mathcal{Q}^{(1)} + \mathcal{Q}^{(2)}$  with  $\mathcal{Q}^{(i)}$  decreasing the filtration index by  $i$ . The cohomology of  $\mathcal{Q}^{(0)}$  is given by (apart from the “trivial” Heisenberg algebra) a Heisenberg algebra represented by three scalar fields. On this algebra, the action of  $\mathcal{Q}^{(1)}$  amounts to the action of three fermionic screenings that satisfy the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  (and the next differential acts trivially). This allows us to show that the Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$  is the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$ .

In Sec. 2, we start with three fermionic screenings in a three-boson realization and impose one relation in the grade  $(1, 1, 1)$  with respect to these operators. This makes three generic fermionic screenings into those representing the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$ . In Sec. 2.2, we explicitly find the lowest-dimension  $\mathcal{WD}(2|1; \alpha)$  operators. In Sec. 2.3, we construct the screenings representing the nilpotent subalgebra of the Langlands-dual quantum group (the sum of three  $\mathcal{U}_q sl(2)$ ). In Sec. 3, we arrive at the same system of three  $\mathcal{U}_q sl(2)$  screenings and the  $W$  algebra in their commutant by “deforming” the  $\widehat{sl}(2)$  WZW theory. This allows us to find two remarkable vertex operators for  $\mathcal{WD}(2|1; \alpha)$  that are analogues of the  $\Phi_{21}$  operators for the Virasoro algebra and also a triplet

operator. These  $\mathcal{WD}(2|1; \alpha)$  vertex operators are used in Sec. 4 to reconstruct the  $\widehat{sl}(2)_{k_i}$  vertex operators and currents and thus to show that  $\mathcal{WD}(2|1; \alpha)$  is indeed the algebra of the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}/\widehat{sl}(2)_{k_1+k_2}$  coset. In Sec. 4.1, we review several relevant points pertaining to vertex-operator extensions, in Sec. 4.2 recall defining properties of the coset, and use these in Sec. 4.3 to reconstruct the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$  currents (and in Sec. 4.4, the corresponding vertex operators) via a vertex-operator extension. The Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$  leading to this  $W$  algebra is worked out in Sec. 5. A number of related issues and further directions are discussed in Sec. 6. Appendix (A) contains an explicit formula for the dimension-4 operator in the commutant of the screenings. In Appendix B, we give our conventions on  $D(2|1; \alpha)$  and the  $\widehat{D}(2|1; \alpha)$  commutation relations.

## 2. FERMIONIC SCREENINGS FOR $\mathcal{WD}(2|1; \alpha)$

In this section, we show that the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  can be generated by three fermionic screenings in the three-boson realization such that the commutant of these screenings is a nontrivial  $W$  algebra.

We recall (see [2, 3]) that  $D(2|1; \alpha)$  is the superalgebra with the bosonic part  $D(2|1; \alpha)_{\overline{0}} = sl(2) \oplus sl(2) \oplus sl(2)$  and the action of  $D(2|1; \alpha)_{\overline{0}}$  on  $D(2|1; \alpha)_{\overline{1}}$  given by the product of the two-dimensional representations. Commutation relations of  $D(2|1; \alpha)$  are written in Appendix B, but in this section, we only use the fact that  $D(2|1; \alpha)$  admits a simple root system where all the three roots are fermionic; the Chevalley generators  $\psi_i$ ,  $i = 1, 2, 3$ , then satisfy  $[\psi_1, \psi_1] = 0$ ,  $[\psi_2, \psi_2] = 0$ ,  $[\psi_3, \psi_3] = 0$  (where  $[\ , \ ]$  denotes the *supercommutator*), and

$$(2.1) \quad [\psi_2, [\psi_1, \psi_3]] + (\alpha + 1)[\psi_3, [\psi_1, \psi_2]] = 0$$

(a non-Serre-type relation in the approach of [9]). Thus, the nilpotent subalgebra contains three even elements  $[\psi_1, \psi_2]$ ,  $[\psi_2, \psi_3]$ , and  $[\psi_3, \psi_1]$  and one more odd element, which up to proportionality is given by any of the triple commutators. The algebra admits an invariant form and can therefore be quantized. Remarkably, the crucial relation in  $\mathcal{U}_q D(2|1; \alpha)$ —the “quantum” analogue of (2.1)—follows from a simple algebraic problem involving three screening operators.

**2.1. A cubic relation on three fermionic screenings.** We consider a formal algebraic problem capturing the commutation properties of screening operators for the  $D(2|1; \alpha)$  root system. We construct three screenings as

$$(2.2) \quad \sigma_i = \int S_i(z), \quad i = 1, 2, 3,$$

where  $S_i$  satisfy formal relations

$$(2.3) \quad S_i(z)S_j(w) = q_{ij} S_j(w)S_i(z), \quad z > w.$$

In the case where  $q_{ij} = q^{a_{ij}}$ , with  $a_{ij}$  being the matrix of scalar products of a simple root system (for the respective range of  $i, j$ ), the corresponding screenings satisfy the relations of the nilpotent

subalgebra of the quantum group determined by the chosen root system.

For  $\sigma_i$  to generate the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  corresponding to three fermionic roots, we take each  $S_i(z)$  to be a fermionic operator and consider the simplest nontrivial relation that can be imposed on all the three operators  $\sigma_i$ . The first nontrivial relation is in the grade  $(1, 1, 1)$  with respect to  $(\sigma_1, \sigma_2, \sigma_3)$ . The only invariant relation in this grade states that the six operators given by the triple integrals of  $S_{p(1)}(z)S_{p(2)}(w)S_{p(3)}(u)$  with  $z < w < u$ , where  $p$  runs over permutations on three elements, are linearly dependent (but no stronger conditions hold). Using (2.3), each of the six operators can be expressed through six “basis” operators  $S_1(z)S_2(w)S_3(u)$  in the respective domains

$$\begin{aligned} z < w < u, \quad w < u < z, \quad u < z < w, \\ z < u < w, \quad w < z < u, \quad u < w < z. \end{aligned}$$

The trilinear relation amounts to the condition that the  $6 \times 6$  matrix relating the two sets of operators must have rank 5. This matrix

$$(2.4) \quad \begin{pmatrix} 1 & q_{23} & q_{12} & q_{12}q_{13} & q_{13}q_{23} & q_{12}q_{13}q_{23} \\ q_{23} & 1 & q_{12}q_{23} & q_{12}q_{13}q_{23} & q_{13} & q_{12}q_{13} \\ q_{12} & q_{12}q_{23} & 1 & q_{13} & q_{13}q_{12}q_{23} & q_{13}q_{23} \\ q_{12}q_{13} & q_{12}q_{23}q_{13} & q_{13} & 1 & q_{12}q_{23} & q_{23} \\ q_{13}q_{23} & q_{13} & q_{12}q_{13}q_{23} & q_{12}q_{23} & 1 & q_{12} \\ q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{13}q_{23} & q_{23} & q_{12} & 1 \end{pmatrix}$$

has the determinant  $(1 - q_{12}^2)^2(1 - q_{13}^2)^2(1 - q_{23}^2)^2(1 - q_{12}^2q_{13}^2q_{23}^2)$ , and we arrive at the condition<sup>4</sup>

$$(2.5) \quad q_{12}^2 q_{13}^2 q_{23}^2 = 1.$$

The relation on  $\sigma_i$  is then given by

$$(2.6) \quad (q_{13}^2 - 1)\sigma_1\sigma_2\sigma_3 + \frac{1 - q_{13}^2q_{23}^2}{q_{23}}\sigma_1\sigma_3\sigma_2 - q_{12}q_{13}^2(1 - q_{23}^2)\sigma_2\sigma_1\sigma_3 + \\ + q_{12}q_{13}(1 - q_{13}^2q_{23}^2)\sigma_2\sigma_3\sigma_1 - \frac{q_{13}(1 - q_{23}^2)}{q_{23}}\sigma_3\sigma_1\sigma_2 + q_{12}q_{13}(q_{13}^2 - 1)q_{23}\sigma_3\sigma_2\sigma_1 = 0.$$

We now realize the  $S_i$  operators in terms of three free fields  $\vec{\varphi} = \{\varphi_1, \varphi_2, \varphi_3\}$  with the operator products

$$(2.7) \quad \partial\varphi_\alpha(z) \partial\varphi_\beta(w) = \frac{\delta_{\alpha\beta}}{(z - w)^2}.$$

Introducing three 3-dimensional vectors  $\vec{a}_i$ , we set (with the Cartesian scalar product)

$$(2.8) \quad S_i = e^{\vec{a}_i \cdot \vec{\varphi}}.$$

The monodromies in (2.3) are then given by

$$(2.9) \quad q_{ij} = e^{\pi i \vec{a}_i \cdot \vec{a}_j}.$$

---

<sup>4</sup>The other vanishings are with the multiplicity 2. We do not consider these “strongly degenerate” cases here, because we are interested in  $D(2|1; \alpha)$ -related structures, but they can also be interesting. For example,  $q_{ij} = -1$  means that the two fermionic screenings point in the opposite directions; the  $\widehat{sl}(2)$  algebra determined by such a pair of fermionic screenings then has the level  $-2$ , which makes a story of its own.



The trilinear relation ensured by (2.5) now reformulates as the condition that the sum of the scalar products be an integer,

$$(2.10) \quad \vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{a}_3 + \vec{a}_2 \cdot \vec{a}_3 = n \in \mathbb{Z}.$$

In accordance with the recipe to construct fermionic screenings, we also have the conditions

$$(2.11) \quad \vec{a}_i \cdot \vec{a}_i = 1, \quad i = 1, 2, 3.$$

To solve the four equations in (2.10) and (2.11) for  $\vec{a}_i$ , we parametrize the scalar products as

$$(2.12) \quad \vec{a}_1 \cdot \vec{a}_2 = k_2 + 1,$$

$$(2.13) \quad \vec{a}_1 \cdot \vec{a}_3 = k_1 + 1,$$

$$(2.14) \quad \vec{a}_2 \cdot \vec{a}_3 = n - 2 - k_1 - k_2$$

in terms of arbitrary parameters  $k_1$  and  $k_2$ . The general solution of Eqs. (2.10)–(2.11) modulo reflections and three-dimensional rotations is then given by<sup>5</sup>

$$(2.15) \quad \vec{a}_1 = \{1, 0, 0\},$$

$$(2.16) \quad \vec{a}_2 = \{k_2 + 1, -i\sqrt{k_2(k_2 + 2)}, 0\},$$

$$(2.17) \quad \vec{a}_3 = \left\{k_1 + 1, -i\frac{2k_2 + k_1(k_2 + 2) - n + 3}{\sqrt{k_2(k_2 + 2)}}, \sqrt{\frac{2(k_2 + 2)k_1^2 + 2(k_2 + 2)(k_2 - n + 3)k_1 + (2k_2 - n + 3)^2}{k_2(k_2 + 2)}}\right\}.$$

Different values of  $n$  correspond to different theories in the commutant of  $(\sigma_1, \sigma_2, \sigma_3)$ . We choose the value corresponding to  $\mathcal{WD}(2|1; \alpha)$  by studying the commutant.

**2.2. The commutant of the screenings.** With  $S_i = e^{\vec{a}_i \cdot \vec{\varphi}}$  constructed in Sec. 2.1, we next look for the commutant of the  $\sigma_i$  operators in the three-boson space. To begin with dimension two, there is one such operator, and it is the energy-momentum tensor

$$(2.18) \quad T = \frac{1}{2}\partial\varphi_1\partial\varphi_1 + \frac{1}{2}\partial\varphi_2\partial\varphi_2 + \frac{1}{2}\partial\varphi_3\partial\varphi_3 - \frac{1}{2}\partial^2\varphi_1 + \frac{i}{2}\sqrt{\frac{k_2}{k_2 + 2}}\partial^2\varphi_2 \\ + \frac{1}{2}\sqrt{\frac{k_2}{k_2 + 2}} \frac{(2k_2 - n + 3)}{\sqrt{2(k_2 + 2)k_1^2 + 2(k_2 + 2)(k_2 - n + 3)k_1 + (2k_2 - n + 3)^2}}\partial^2\varphi_3.$$

Its central charge is given by

$$(2.19) \quad c = \frac{3k_1k_2(k_1 + k_2 - n + 3)}{2(k_2 + 2)k_1^2 + 2(k_2 + 2)(k_2 - n + 3)k_1 + (2k_2 - n + 3)^2}.$$

Searching for higher-dimension primary operators in the commutant, we find that a symmetry enhancement occurs for  $n = -1$ , when the commutant involves a primary operator with the minimum possible dimension  $> 2$ , i.e., dimension 4.

**Lemma 2.1.** *For generic  $k_1$  and  $k_2$ , an operator of conformal dimension 4 that commutes with the screenings  $\sigma_i$  determined by Eqs. (2.2), (2.8), and (2.10)–(2.11) and is primary with respect to the*

<sup>5</sup>The imaginary unit means that the signature of the three-dimensional space of the scalars is  $(+, -, +)$ , i.e.,  $\varphi_2$  could be redefined such that  $\partial\varphi_2(z)\partial\varphi_2(w) = -1/(z - w)^2$  and  $i$  consistently removed.

energy-momentum tensor (2.18) exists if and only if  $n = -1$ .

This is shown by solving the system of equations ensuring that a general dimension-4 operator commutes with the three fermionic screenings (there are 51 potentially possible operator terms constructed out of the free fields, with two terms corresponding to descendants of the energy-momentum tensor). As a function of  $n$ , the determinant of the system vanishes if and only if  $n = -1$  (for generic  $k_1$  and  $k_2$ ), and the corresponding solution gives the operator written in Appendix A.

We thus consider the  $n = -1$  case in the above theory. The energy-momentum tensor  $T|_{n=-1}$  (which we denote simply by  $T$  in what follows),

$$(2.20) \quad T = \frac{1}{2}\partial\varphi_1\partial\varphi_1 + \frac{1}{2}\partial\varphi_2\partial\varphi_2 + \frac{1}{2}\partial\varphi_3\partial\varphi_3 - \frac{1}{2}\partial^2\varphi_1 \\ + \frac{i\sqrt{k_2}}{2\sqrt{k_2+2}}\partial^2\varphi_2 - \frac{\sqrt{k_2}}{\sqrt{2(k_1+2)(k_1+k_2+2)}}\partial^2\varphi_3,$$

has the central charge given by Eq. (1.4).

We define the  $W$  algebra  $\mathcal{WD}(2|1;\alpha)$  to be the commutant of the operators  $\sigma_i = \oint e^{\vec{a}_i \cdot \vec{\varphi}}$  with  $\vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{a}_3 + \vec{a}_2 \cdot \vec{a}_3 = -1$  and  $\vec{a}_i \cdot \vec{a}_i = 1$  that generate the nilpotent subalgebra of  $\mathcal{U}_q D(2|1;\alpha)$ .

For  $n = -1$ , we now summarize the screenings that we use in what follows:

$$(2.21) \quad \sigma_1 = \oint e^{\vec{a}_1 \cdot \vec{\varphi}}, \quad \vec{a}_1 = \{1, 0, 0\},$$

$$(2.22) \quad \sigma_2 = \oint e^{\vec{a}_2 \cdot \vec{\varphi}}, \quad \vec{a}_2 = \{k_2 + 1, -i\sqrt{k_2(k_2 + 2)}, 0\},$$

$$(2.23) \quad \sigma_3 = \oint e^{\vec{a}_3 \cdot \vec{\varphi}}, \quad \vec{a}_3 = \{k_1 + 1, -i\frac{(k_1+2)\sqrt{k_2+2}}{\sqrt{k_2}}, \sqrt{\frac{2(k_1+2)(k_1+k_2+2)}{k_2}}\}.$$

**Remark 2.2.** The semiclassical limit of the trilinear relation on  $\sigma_i$  is the  $D(2|1;\alpha)$  relation (2.1). Indeed, inserting (2.9) with the above  $\vec{a}_i$  in Eq. (2.6), we obtain

$$(2.24) \quad (e^{2\pi i k_1} - 1)\sigma_1\sigma_2\sigma_3 - (e^{\pi i(k_1+k_2)} - e^{\pi i(k_1-k_2)})\sigma_1\sigma_3\sigma_2 + (e^{\pi i(2k_1+k_2)} - e^{-\pi i k_2})\sigma_2\sigma_1\sigma_3 + \\ + (e^{\pi i(k_1+k_2)} - e^{i\pi(k_1-k_2)})\sigma_2\sigma_3\sigma_1 - (e^{\pi i(2k_1+k_2)} - e^{-\pi i k_2})\sigma_3\sigma_1\sigma_2 - (e^{2\pi i k_1} - 1)\sigma_3\sigma_2\sigma_1 = 0.$$

Semiclassically, in the first nontrivial order, this becomes

$$(2.25) \quad k_1[[\sigma_1, \sigma_2], \sigma_3] + k_2[\sigma_2, [\sigma_1, \sigma_3]] = 0.$$

Comparing (2.25) and (2.1), we conclude that in the classical limit, the algebra generated by  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  is the nilpotent subalgebra of  $D(2|1;\alpha)$  with (using the identifications  $\sigma_1 = \psi_1$ ,  $\sigma_2 = \psi_3$ , and  $\sigma_3 = \psi_2$ )

$$(2.26) \quad \alpha = -1 - \frac{k_2}{k_1}.$$

The “quantum” relation (2.24) is the corresponding quantum-group deformation of (2.1).

**2.3. Bosonic screenings.** Any chosen pair of the fermionic screenings  $\{\sigma_i, \sigma_j\}$ ,  $i \neq j$ , determines (the nilpotent subalgebra of) the quantum group  $\mathcal{U}_q sl(2|1)$  (with the simple roots of  $sl(2|1)$  chosen fermionic). Accordingly, the commutant of these two screenings in the appropriate two-boson subspace is the  $W$  algebra  $\mathcal{W}sl(2|1)$ . This algebra commutes with a third, bosonic, screening operator of a *non-vertex-operator* form (the product of a current and an exponential). The occurrence of this screening can be seen by invoking the  $\widehat{sl}(2)$  argument: the commutant of any *pair* of fermionic screenings in the three-boson realization is (the symmetric realization of) the  $\widehat{sl}(2)$  algebra, where the occurrence of the third screening, of a non-vertex-operator form, is well known. The  $\mathcal{WD}(2|1; \alpha)$  algebra is the intersection of the three algebras of the form  $(\mathcal{W}sl(2|1) \otimes (\text{Heisenberg}))$ . Therefore, the commutant of  $\mathcal{WD}(2|1; \alpha)$  contains the bosonic screenings

$$(2.27) \quad \rho_{ij} = \oint R_{ij}, \quad R_{ij} = J_{ij} e^{\vec{r}_{ij} \cdot \vec{\varphi}}, \quad i \neq j,$$

where

$$(2.28) \quad \vec{r}_{ij} = -\frac{1}{(\vec{a}_i \cdot \vec{a}_j) + 1}(\vec{a}_i + \vec{a}_j)$$

and  $J_{ij}$  is a linear combination of  $\vec{a}_i \cdot \partial \vec{\varphi}$  and  $\vec{a}_j \cdot \partial \vec{\varphi}$ . By adding a total derivative to  $R_{ij}$  and by overall normalizations, the coefficients in this linear combination can be fixed arbitrarily.

The vectors  $\vec{r}_{ij}$  satisfy

$$(2.29) \quad (\vec{r}_{12})^2 = \frac{2}{k_2 + 2}, \quad (\vec{r}_{13})^2 = \frac{2}{k_1 + 2}, \quad (\vec{r}_{23})^2 = \frac{-2}{k_1 + k_2 + 2}.$$

As another consequence of setting  $n = -1$  in (2.17), we have

$$(2.30) \quad \vec{r}_{12} \cdot \vec{r}_{13} = 0, \quad \vec{r}_{12} \cdot \vec{r}_{23} = 0, \quad \vec{r}_{23} \cdot \vec{r}_{13} = 0$$

and (with  $\vec{r}_{ij} = \vec{r}_{ji}$  for  $i \neq j$ )

$$(2.31) \quad \begin{aligned} \vec{a}_i \cdot \vec{r}_{jk} &= 1, \quad i, j, k \text{ all distinct,} \\ \vec{a}_i \cdot \vec{r}_{ij} &= -1. \end{aligned}$$

The following statement is now readily verified.

**Lemma 2.3.** *The operators  $\rho_{12}$ ,  $\rho_{13}$ , and  $\rho_{23}$  pairwise commute. Each  $\rho_{ij}$  commutes with  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ .*

Explicitly, the integrands of the bosonic screenings are given by

$$(2.32) \quad R_{12} = (\alpha_{12} \partial \varphi_1 + i \beta_{12} \partial \varphi_2) e^{-\varphi_1 + i \sqrt{\frac{k_2}{k_2+2}} \varphi_2},$$

$$(2.33) \quad \begin{aligned} R_{13} = & \left( (\alpha_{13} + \beta_{13}(k_1 + 1)) \partial \varphi_1 - i \beta_{13}(k_1 + 2) \sqrt{\frac{k_2+2}{k_2}} \partial \varphi_2 \right. \\ & \left. + \sqrt{2} \beta_{13} \sqrt{\frac{(k_1+2)(k_1+k_2+2)}{k_2}} \partial \varphi_3 \right) e^{-\varphi_1 + i \sqrt{\frac{k_2+2}{k_2}} \varphi_2 - \sqrt{\frac{2(k_1+k_2+2)}{k_2(k_1+2)}} \varphi_3}, \end{aligned}$$

$$(2.34) \quad R_{23} = \left( (\alpha_{23}(k_2 + 1) + \beta_{23}(k_1 + 1)) \partial \varphi_1 - i(\beta_{23}(k_1 + 2) + \alpha_{23}k_2) \sqrt{\frac{k_2+2}{k_2}} \partial \varphi_2 \right. \\ \left. + \sqrt{2} \beta_{23} \sqrt{\frac{(k_1+2)(k_1+k_2+2)}{k_2}} \partial \varphi_3 \right) e^{\varphi_1 - i\sqrt{\frac{k_2+2}{k_2}} \varphi_2 + \sqrt{\frac{2(k_1+2)}{k_2(k_1+k_2+2)}} \varphi_3}.$$

In the next section, these operators are reproduced from a different argument with fixed values of the  $\alpha_{ij}$  and  $\beta_{ij}$  coefficients.

Thus, we have constructed screening operators that generate the nilpotent subalgebras of the quantum groups  $\mathcal{U}_q D(2|1; \alpha)$  and  $\mathcal{U}_{q_1} sl(2) \otimes \mathcal{U}_{q_2} sl(2) \otimes \mathcal{U}_{q_3} sl(2)$  (with the quantum-group parameters that are explicitly written in what follows). In the next section, we construct  $\mathcal{WD}(2|1; \alpha)$  vertex operators carrying representations of the  $\mathcal{U}_{q_i} sl(2)$  quantum groups.

### 3. $\mathcal{WD}(2|1; \alpha)$ BY DEFORMATION AND THE $\mathcal{WD}(2|1; \alpha)$ VERTEX OPERATORS

In this section, we show that the  $\mathcal{WD}(2|1; \alpha)$  algebra can be constructed by deforming (a subalgebra of) the  $\widehat{\mathcal{U}sl}(2)$  algebra.<sup>6</sup> This allows us to construct  $\mathcal{WD}(2|1; \alpha)$  vertex operators, in particular those that are 212, 122, 221, and 113 representations of the three  $\mathcal{U}_q sl(2)$  quantum groups corresponding to the  $\rho_{ij}$  screenings. The operators that we need in the next sections are summarized in (3.40).

The outline of this section is as follows. In Sec. 3.1, we deform  $\widehat{\mathcal{U}sl}(2)$  into a  $W$  algebra by taking the commutant of two bosonic screenings. Together with the standard Wakimoto screening, the bosonic screenings generate three commuting  $\mathcal{U}_q sl(2)$  quantum groups. In Sec. 3.2, we construct vertex operators that are singlets with respect to one of these quantum groups and are doublets with respect to the other two. In Sec. 3.3, we use the “*symmetric*” bosonization, which allows us to additionally construct the vertex operator that is a triplet with respect to one of the three  $sl(2)$  quantum groups. We then show that the  $W$  algebra obtained by deforming  $\widehat{\mathcal{U}sl}(2)$  is in fact  $\mathcal{WD}(2|1; \alpha)$  and explicitly map the “deformed” picture onto that in Sec. 2.

**3.1. A “ $\beta$ -deformation” of  $\widehat{sl}(2)$ .** We start with the level- $k$   $\widehat{sl}(2)$  algebra, which in terms of operator products is given by

$$(3.1) \quad \begin{aligned} J^+(z) J^-(w) &= \frac{k}{(z-w)^2} + \frac{2J^0}{z-w}, \\ J^0(z) J^\pm(w) &= \frac{\pm J^\pm}{z-w}, \quad J^0(z) J^0(w) = \frac{k/2}{(z-w)^2}, \end{aligned}$$

and consider the subalgebra in  $\widehat{\mathcal{U}sl}(2)$  that commutes with the *Lie* algebra  $sl(2)$ . We next deform this vertex operator algebra using the operators

$$(3.2) \quad \sigma^\pm = \oint S^\pm, \quad S^\pm(z) = J^\pm(z) e^{\beta^\pm \int J^0}$$

---

<sup>6</sup>By  $\mathcal{U}$ , we here mean the vertex operator algebra and sometimes, its vacuum representation (this must not lead to a confusion).

with the two parameters  $\beta^+$  and  $\beta^-$  related by

$$(3.3) \quad \beta^\mp = \frac{\beta^\pm}{1 \pm \frac{k}{2}\beta^\pm},$$

which implies that

$$(3.4) \quad [\sigma^+, \sigma^-] = 0.$$

We let  $W_*$  denote the commutant of  $\sigma^+$  and  $\sigma^-$  in  $\widehat{\mathcal{U}sl}(2)$ .

We write  $\beta \equiv \beta^+$  and  $\beta' \equiv \beta^-$  in what follows; thus,

$$(3.5) \quad \beta' = \frac{\beta}{1 + \frac{k}{2}\beta}.$$

We next introduce the Wakimoto bosonization

$$(3.6) \quad J^+ = -a,$$

$$(3.7) \quad J^0 = aa^* + \sqrt{\frac{k+2}{2}} \partial\varphi,$$

$$(3.8) \quad J^- = (a^*)^2 a + \sqrt{2(k+2)} a^* \partial\varphi + k \partial a^*$$

and additionally represent the bosonic ghosts  $a$  and  $a^*$  (with the operator product  $a(z)a^*(w) = \frac{-1}{z-w}$ ) in terms of free scalars  $f$  and  $\phi$  with  $\partial f(z)\partial f(w) = \frac{1}{(z-w)^2}$  and  $\partial\phi(z)\partial\phi(w) = \frac{-1}{(z-w)^2}$  via

$$(3.9) \quad a = -e^{-f-\phi}, \quad a^* = \partial f e^{f+\phi}.$$

This gives a three-boson realization of  $\widehat{sl}(2)$  (known as the standard or the asymmetric realization). Normal ordering of all composite operators is understood. We also recall the Wakimoto bosonization screening

$$(3.10) \quad \sigma_W = \oint S_W, \quad S_W = -a e^{-\sqrt{\frac{2}{k+2}}\varphi}.$$

The  $W_*$  algebra is now selected from the free-field space as the commutant of  $\sigma_W$ ,  $\sigma^+$ , and  $\sigma^-$  (all of which are called screenings hereafter). By a direct calculation, we obtain

**Lemma 3.1.** *The algebra  $W_*$  contains a Virasoro algebra with the central charge*

$$(3.11) \quad c_\perp = \frac{3k}{k+2} - \frac{3k\beta^2(k\beta+4)^2}{2(k\beta+2)^2}.$$

For the “undeformed” value  $\beta = 0$ , this is indeed the  $\widehat{sl}(2)$  WZW model central charge. In the calculation, we used that the operators in the integrands of  $\sigma^\pm$  rewrite as

$$(3.12) \quad J^+(z) e^{\beta \int \tilde{J}^0} = e^{(\beta-1)\phi} e^{\beta \sqrt{\frac{k+2}{2}}\varphi} e^{-f}(z)$$

and

$$(3.13) \quad J^-(z) e^{\beta' \int \tilde{J}^0} = \left( (k+1)\partial^2 f + (k+1)\partial f \partial f + (k+2)\partial f \partial \phi \right. \\ \left. + \sqrt{2(k+2)}\partial\varphi \partial f \right) e^{(\beta'+1)\phi} e^{\beta' \sqrt{\frac{k+2}{2}}\varphi} e^f(z).$$

**3.2. 122-type quantum group representations.** We next recall the spin- $\frac{1}{2}$  vertex operator of  $\widehat{sl}(2)_k$  and deform it into a certain vertex operator for  $W_*$ . The spin- $\frac{1}{2}$  vertex is a 2-dimensional quantum group representation with the highest-weight vector given by

$$(3.14) \quad \Psi \equiv \Phi_{\frac{1}{2}} = e^{\frac{1}{\sqrt{2(k+2)}}\varphi}.$$

The action of the horizontal  $sl(2)$  subalgebra on  $\Psi$  generates a two-dimensional representation,

$$(3.15) \quad J_0^- \Psi = a^* e^{\frac{1}{\sqrt{2(k+2)}}\varphi}, \quad (J_0^-)^2 \Psi = 0, \quad J_0^+ \Psi = 0.$$

These properties persist under the “deformation” of  $J^\pm$  by  $e^{\beta^\pm \int^z J^0}$  if the vertex is deformed into

$$(3.16) \quad \Psi_\beta(w) = \Psi(w) e^{-\frac{1}{2}\beta' \int^w J^0} = e^{-\frac{1}{2}\beta' \phi} e^{\left(\frac{1}{\sqrt{2(k+2)}} - \frac{\beta'}{2} \sqrt{\frac{k+2}{2}}\right)\varphi}(w).$$

(we recall that  $\beta'$  is expressed through  $\beta$  and the level  $k$ ). Namely, we have

$$(3.17) \quad J^-(z) e^{\beta' \int^z J^0} \cdot \Psi_\beta(w) = (z-w)^{(\beta'+1)\frac{\beta'}{2} + \beta'(-\frac{k+2}{4}\beta' + \frac{1}{2})-1} \partial f e^{(\frac{1}{2}\beta'+1)\phi} e^{\frac{\frac{1}{2}\beta'(k+2)+1}{\sqrt{2(k+2)}}\varphi} e^f(w).$$

Remarkable cancellations of poles occur in acting with (3.13) on the right-hand side of the last formula, resulting in

$$(3.18) \quad J^-(z) e^{\beta' \int^z J^0} \cdot \partial f e^{(\frac{1}{2}\beta'+1)\phi} e^{\frac{\frac{1}{2}\beta'(k+2)+1}{\sqrt{2(k+2)}}\varphi} e^f(w) \sim (z-w)^{-(\beta'+1)(\frac{1}{2}\beta'+1) + \beta'(\frac{k+2}{4}\beta' + \frac{1}{2})+1}.$$

Two effects now occur. First,  $(\sigma^-)^2 \Psi_\beta(w)$  is a local operator, because the sum of the exponents in (3.17) and (3.18) is an integer,

$$(3.19) \quad [(\beta'+1)\frac{\beta'}{2} + \beta'(-\frac{k+2}{4}\beta' + \frac{1}{2}) - 1] + [-(\beta'+1)(\frac{1}{2}\beta'+1) + \beta'(\frac{k+2}{4}\beta' + \frac{1}{2}) + 1] = -1.$$

Second, moreover, the relevant first-order pole vanishes (one extra positive power comes from the integration), and therefore, applying the standard argument, we conclude that

$$(3.20) \quad (\sigma^-)^2 \Psi_\beta(w) = 0.$$

Thus,  $\Psi_\beta$  is a part of a doublet under the action of  $\sigma^-$ . At the same time, it is a singlet with respect to  $\sigma^+$ . Further, the spin- $\frac{1}{2}$  vertex operator is a doublet under the action of the Wakimoto screening, and because the deformation involves only the  $\widehat{sl}(2)$  current  $J^0$ , this property persists for  $\Psi_\beta$ . We thus conclude that  $\Psi_\beta$  is the  $(1, 2, 2)$  representation of (the quantum  $sl(2)$  groups with the respective upper-triangular generators)  $(\sigma^+, \sigma_W, \sigma^-)$ .

A similar argument shows that the operator

$$(3.21) \quad \Psi'_\beta(z) = a^*(z) \Psi(z) e^{-\frac{1}{2}\beta \int^z J^0} = e^{f + (1-\frac{1}{2}\beta)\phi - \frac{(k+2)\beta-2}{2\sqrt{2(k+2)}}\varphi}(z)$$

is the  $(2, 2, 1)$  representation of (the quantum  $sl(2)$  groups with the respective upper-triangular generators)  $(\sigma^+, \sigma_W, \sigma^-)$ .

**3.3. Symmetric bosonization and the triplet operator.** The above results do not depend on a particular bosonization of the  $aa^*$  system; from now on, we use the alternative bosonization where

$$(3.22) \quad a = -\partial f e^{-f-\phi}, \quad a^* = e^{f+\phi}$$

(for the  $\widehat{sl}(2)$  algebra, this gives the so-called *symmetric* three-boson realization). The point is that *all the three operators*  $(S^+, S^-, S_W)$  then take the form (current)  $\cdot$  (exponential),

$$(3.23) \quad S^+ = \partial f e^{(\beta-1)\phi} e^{\beta\sqrt{\frac{k+2}{2}}\varphi} e^{-f},$$

$$(3.24) \quad S_W = \partial f e^{-\phi} e^{-\sqrt{\frac{2}{k+2}}\varphi} e^{-f},$$

$$(3.25) \quad S^- = \left( (k+1)\partial f + (k+2)\partial\phi + \sqrt{2(k+2)}\partial\varphi \right) e^{(\beta'+1)\phi} e^{\beta'\sqrt{\frac{k+2}{2}}\varphi} e^f.$$

In addition, it is straightforward to show that the operator

$$(3.26) \quad \Upsilon_\beta = (4(\beta-1)\partial\phi + 2\sqrt{2(k+2)}\beta\partial\varphi + (k\beta^2 + 4\beta - 4)\partial f) e^{\phi + \sqrt{\frac{2}{k+2}}\varphi + f}$$

is the  $(1, 3, 1)$  representation of the quantum  $sl(2)$  groups with the respective upper-triangular generators  $(\sigma^+, \sigma_W, \sigma^-)$ .

We note that in this bosonization, the energy-momentum tensor with central charge (3.11) in the commutant of the screenings is explicitly given by

$$(3.27) \quad T_\perp = \frac{1}{2}\partial f \partial f - \frac{1}{2}\partial\phi \partial\phi + \frac{1}{2}\partial\varphi \partial\varphi - \frac{1}{2}\partial^2 f + \frac{k\beta^2 - (k-4)\beta - 2}{2k\beta + 4}\partial^2\phi + \frac{k(k+2)\beta^2 + 2(k+4)\beta - 4}{\sqrt{2(k+2)}(2k\beta + 4)}\partial^2\varphi.$$

Using the symmetric bosonization, we next identify the above screenings with the bosonic screenings constructed in Sec. 2, which also are of the form (current)  $\cdot$  (exponential).

**3.3.1. Mapping onto  $(\varphi_1, \varphi_2, \varphi_3)$ .** Writing  $S^+ = (\dots)e^{\vec{r}^+ \cdot \vec{\Phi}}$ ,  $S_W = (\dots)e^{\vec{r}_W \cdot \vec{\Phi}}$ , and  $S^- = (\dots)e^{\vec{r}^- \cdot \vec{\Phi}}$ , where the dots denote currents and  $\vec{\Phi} = (\phi, \varphi, f)$ , we note that the three 3-dimensional vectors in the exponents are pairwise orthogonal and satisfy

$$(3.28) \quad \vec{r}^+ \cdot \vec{r}^+ = \frac{1}{2}\beta(k\beta + 4), \quad \vec{r}_W \cdot \vec{r}_W = \frac{2}{k+2}, \quad \vec{r}^+ \cdot \vec{r}^+ = \frac{1}{2}\beta'(k\beta' - 4) = -\frac{2\beta(k\beta + 4)}{(k\beta + 2)^2}.$$

The identification between the screenings in (3.23)–(3.25) and those in Sec. 2.3 can be chosen such that

$$(3.29) \quad \vec{r}^+ \cdot \vec{r}^+ = \vec{r}_{13} \cdot \vec{r}_{13}, \quad \vec{r}^- \cdot \vec{r}^- = \vec{r}_{12} \cdot \vec{r}_{12}.$$

These two equations imply

$$(3.30) \quad k = -k_1 - k_2 - 4,$$

$$(3.31) \quad \beta = \frac{2}{k_1 + 2 + \sqrt{-(k_1 + 2)(k_2 + 2)}}.$$

With these  $k$  and  $\beta$ , the central charge in Eq. (3.11) becomes the one in Eq. (1.4). Further, equating the respective exponentials in

$$(3.32) \quad R_{13} = S^+, \quad R_{23} = S_W, \quad R_{12} = S^-,$$

we obtain a system of three equations that is solved by

$$\begin{aligned}
\partial\phi &= -\frac{2\sqrt{-(k_1+2)(k_2+2)}+(k_1+2)(k_1+k_2+2)}{k_1+k_2+4}\partial\varphi_1 + i\frac{2(k_2+1)\sqrt{-(k_1+2)(k_2+2)}+(k_1+3)(k_2+2)(k_1+k_2+2)}{\sqrt{k_2}\sqrt{k_2+2}(k_1+k_2+4)}\partial\varphi_2 \\
&\quad -\frac{\sqrt{2(k_1+k_2+2)}(\sqrt{-(k_1+2)(k_2+2)}+(k_1+2)(k_1+k_2+3))}{\sqrt{k_1+2}\sqrt{k_2}(k_1+k_2+4)}\partial\varphi_3, \\
\partial\varphi &= -i\frac{\sqrt{2(k_1+k_2+2)}(k_1+2+\sqrt{-(k_1+2)(k_2+2)})}{k_1-k_2+2\sqrt{-(k_1+2)(k_2+2)}}\partial\varphi_1 -\frac{\sqrt{2(k_1+k_2+2)}((k_1+1)(k_2+2)+\sqrt{-(k_1+2)(k_2+2)}(k_2+3))}{\sqrt{k_2}\sqrt{k_2+2}(k_1-k_2+2\sqrt{-(k_1+2)(k_2+2)})}\partial\varphi_2 \\
&\quad +i\frac{\sqrt{-(k_1+2)(k_2+2)}(k_1+k_2+2)-2(k_1+2)(k_1+k_2+3)}{\sqrt{k_1+2}\sqrt{k_2}(k_1+k_2+4)}\partial\varphi_3, \\
\partial f &= (k_1+1)\partial\varphi_1 -i(k_1+2)\sqrt{\frac{k_2+2}{k_2}}\partial\varphi_2 +\frac{\sqrt{2(k_1+2)}\sqrt{k_1+k_2+2}}{\sqrt{k_2}}\partial\varphi_3
\end{aligned}$$

In terms of the three currents introduced in Sec. 2, the integrands of the screenings become

$$(3.33) \quad R_{13} = S^+ = \vec{a}_3 \cdot \partial\vec{\varphi} e^{\vec{r}_{13} \cdot \vec{\varphi}},$$

$$(3.34) \quad R_{23} = S_W = -\vec{a}_3 \cdot \partial\vec{\varphi} e^{\vec{r}_{23} \cdot \vec{\varphi}},$$

$$(3.35) \quad R_{12} = S^- = \vec{a}_2 \cdot \partial\vec{\varphi} e^{\vec{r}_{12} \cdot \vec{\varphi}}$$

and the “deformed” vertex operators are mapped into

$$(3.36) \quad \Psi_\beta = e^{-\frac{1}{2}(\vec{r}_{12}+\vec{r}_{23}) \cdot \vec{\varphi}},$$

$$(3.37) \quad \Psi'_\beta = e^{-\frac{1}{2}(\vec{r}_{13}+\vec{r}_{23}) \cdot \vec{\varphi}}.$$

Their dimensions evaluated with respect to the energy-momentum tensor in Eq. (3.27) are expressed in terms of  $k_1$  and  $k_2$  as

$$(3.38) \quad \dim \Psi_\beta = \frac{3k_1}{4(k_2+2)(k_1+k_2+2)}, \quad \dim \Psi'_\beta = \frac{3k_2}{4(k_1+2)(k_1+k_2+2)}.$$

The  $\Upsilon_\beta$  operator in Eq. (3.26) becomes, up to normalization,

$$(3.39) \quad \Upsilon_\beta = \vec{a}_1 \cdot \partial\vec{\varphi} e^{-\vec{r}_{23} \cdot \vec{\varphi}}.$$

We note that its property to be the  $(1, 1, 3)$  representation with respect to  $(\rho_{12}, \rho_{13}, \rho_{23})$  is now readily checked using (2.31).

**3.3.2. Identification with  $\mathcal{WD}(2|1; \alpha)$ .** We now identify the  $W_*$  algebra in the commutant of the  $(\sigma^+, \sigma_W, \sigma^-)$  screenings with  $\mathcal{WD}(2|1; \alpha)$ . In the symmetric bosonization of  $\widehat{sl}(2)$ , there is one bosonic (Wakimoto) and two fermionic screenings. The  $W_*$  algebra is therefore selected from the free-field space by five screening operators. This set of screenings is redundant: as we have seen,  $W_*$  is in fact the commutant of  $\sigma^+$ ,  $\sigma_W$ , and  $\sigma^-$ . On the other hand, it is the commutant of the two fermionic screenings and  $\sigma^+$  (equivalently, of the two fermionic screenings and  $\sigma^-$ ). This follows from viewing  $W_*$  as the intersection of two algebras of the form  $\mathcal{Wsl}(2|1) \otimes (\text{Heisenberg})$ . These three screenings, however, have been identified with the respective screenings in Sec. 2. It only remains to verify the identification for the fermionic screenings in the symmetric bosonization



of  $\widehat{sl}(2)$ ,

$$e^{(k+2)\phi + \sqrt{2(k+2)}\varphi + (k+1)f} \quad \text{and} \quad e^f.$$

Under the mapping in Sec. 3.3.1, these are indeed mapped into  $\sigma_2$  and  $\sigma_3$  in Eqs. (2.22)–(2.23). We thus conclude that the  $W$  algebra in the commutant of  $(\sigma^+, \sigma_W, \sigma^-)$  coincides with  $\mathcal{WD}(2|1; \alpha)$ .

Thus, we have constructed two vertex operators for  $\mathcal{WD}(2|1; \alpha)$  such that each vertex is a singlet with respect to one of the three  $\mathcal{U}_q sl(2)$  quantum groups and is (the highest-weight vector in) a doublet with respect to each of the other two. These operators are used in reconstructing the spin- $\frac{1}{2}$  vertex operators of  $\widehat{sl}(2)_{k_1}$  and  $\widehat{sl}(2)_{k_2}$ . In reconstructing the currents, we also use the operator in Eq. (3.39). We change the notation for these operators as

$$(3.40) \quad \Psi_{212} = e^{-\frac{1}{2}(\vec{r}_{12} + \vec{r}_{23}) \cdot \vec{\varphi}}, \quad \Psi_{122} = e^{-\frac{1}{2}(\vec{r}_{13} + \vec{r}_{23}) \cdot \vec{\varphi}}, \quad \Upsilon_{113} = \vec{a}_1 \cdot \partial \vec{\varphi} e^{-\vec{r}_{23} \cdot \vec{\varphi}},$$

where the subscripts refer to representations of the  $\mathcal{U}_q sl(2)$  quantum groups with the respective upper-triangular generators  $\rho_{12}$ ,  $\rho_{13}$ , and  $\rho_{23}$ .

#### 4. THE VERTEX OPERATOR RECONSTRUCTION

In this section, we reconstruct  $\widehat{sl}(2)_{k_1}$  and  $\widehat{sl}(2)_{k_2}$  currents and vertex operators using  $\mathcal{WD}(2|1; \alpha)$  vertex operators. This will imply that  $\mathcal{WD}(2|1; \alpha)$  is the  $W$  algebra of the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$  coset. Because we use the language of vertex-operator extensions, we start with reviewing several relevant points in Sec. 4.1. In Sec. 4.2, we recall defining properties of the  $\widehat{sl}(2) \oplus \widehat{sl}(2) / \widehat{sl}(2)$  coset theory (originating in [10, 11], see also [12]) and then show that these can be recovered starting from the  $\mathcal{WD}(2|1; \alpha)$  algebra. In Sec. 4.3, we reconstruct the  $\widehat{sl}(2)_{k_1}$  and  $\widehat{sl}(2)_{k_2}$  currents, and in Sec. 4.4, vertex operators.

**4.1. Quantum groups and vertex-operator extensions.** Vertex-operator extensions  $\mathcal{A} \rightarrow \mathcal{B}$ , e.g., the one that we consider in (4.20) in what follows, are constructed by adding *local* fields to the algebra  $\mathcal{A}$ ; such fields are selected as those vertex operators of  $\mathcal{A}$  that have trivial monodromies with respect to each other. Because the monodromy properties are encoded in quantum group representations, the procedure involves taking quantum-group singlets that are simply transposed under the action of the corresponding  $R$ -matrix. For the particular type of vertex-operator extensions that we use, the  $\mathcal{A}$  algebra is taken to be a product  $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ , where each algebra  $\mathcal{A}^{(i)}$  is considered together with an algebra of its vertex operators; from the  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  vertex operators, one then constructs all possible *local* fields (i.e., fields with trivial monodromies with respect to each other). A commonly used recipe to construct local fields with the help of  $R$  matrices amounts to finding vertex operators  $V_\alpha^{(i)}$  of the respective algebra  $\mathcal{A}^{(i)}$  and the elements

$$(4.1) \quad w_{\alpha\alpha'} \in V_\alpha^{(1)} \otimes V_{\alpha'}^{(2)}$$

(where we somewhat abuse the notation by identifying vertex operators with the corresponding quantum-group representation spaces) such that the monodromy properties

$$(4.2) \quad R_{\alpha\beta}^{(i)} : V_\alpha^{(i)} \otimes V_\beta^{(i)} \rightarrow V_\beta^{(i)} \otimes V_\alpha^{(i)}$$

imply that the mapping

$$(4.3) \quad P_{23} \circ R_{\alpha\beta}^{(1)} \otimes R_{\alpha'\beta'}^{(2)} \circ P_{23} : V_\alpha^{(1)} \otimes V_{\alpha'}^{(2)} \otimes V_\beta^{(1)} \otimes V_{\beta'}^{(2)} \rightarrow V_\beta^{(1)} \otimes V_{\beta'}^{(2)} \otimes V_\alpha^{(1)} \otimes V_{\alpha'}^{(2)},$$

reduces to transpositions of the chosen elements,

$$(4.4) \quad P_{23} \circ R_{\alpha\beta}^{(1)} \otimes R_{\alpha'\beta'}^{(2)} \circ P_{23} : w_{\alpha\alpha'} \otimes w_{\beta\beta'} \mapsto w_{\beta\beta'} \otimes w_{\alpha\alpha'}.$$

All such operators  $w_{\alpha\alpha'}$  then make up a local algebra, which is a vertex-operator extension  $\mathcal{B}$  of  $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ . Next, (some of) the *vertex operators* of  $\mathcal{B}$  can be constructed similarly, by combining  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  vertex operators into operators that have some prescribed monodromy properties instead of the trivial monodromy in (4.4) (and are local with respect to the elements of  $\mathcal{B}$ ).

In what follows, we let  $\mathbb{C}_q^n$  denote the  $n$ -dimensional module over the quantum group  $\mathcal{U}_q \mathfrak{sl}(2)$  (we use these with  $n = 2$  and  $3$ ); abusing the notation, we also write  $\mathbb{C}_\kappa^2$  whenever  $q = e^{\pi i \kappa}$ . The relevant value of the  $q$  parameter for each quantum group is read off from the corresponding screening operator. We recall from Sec. 3.3.1 the vertex operators  $\Psi_{212}$  and  $\Psi_{122}$  that are the  $(2, 1, 2)$  and  $(1, 2, 2)$  representations of the  $\mathcal{U}_q \mathfrak{sl}(2)$  quantum groups with the upper-triangular generators  $\rho_{12}$ ,  $\rho_{13}$ , and  $\rho_{23}$ . It follows from the formulae of Sec. 2 that the respective quantum group parameters are  $q_j = e^{\pi i \kappa_j}$  with

$$(4.5) \quad \kappa_1 = \frac{2}{k_2 + 2}, \quad \kappa_2 = \frac{2}{k_1 + 2}, \quad \kappa_3 = \frac{-2}{k_1 + k_2 + 2}.$$

Abusing the terminology, we often extend the notations  $\Psi_{212}$  and  $\Psi_{122}$  to the respective 4-dimensional quantum-group representations, with the specific operators in the right-hand sides of (3.40) being the highest-weight vectors in these representations. The  $(2, 1, 2)$  representation content of  $\Psi_{212}$  can therefore be expressed as

$$(4.6) \quad \Psi_{212}(z) \in (\mathbb{C}_{\frac{2}{k_2+2}}^2 \otimes \mathbb{C}_{-\frac{2}{k_1+k_2+2}}^2)(z)$$

and likewise,

$$(4.7) \quad \Psi_{122}(z) \in (\mathbb{C}_{\frac{2}{k_1+2}}^2 \otimes \mathbb{C}_{-\frac{2}{k_1+k_2+2}}^2)(z).$$

Similarly, we also write

$$(4.8) \quad \Upsilon_{113}(z) \in \mathbb{C}_{-\frac{2}{k_1+k_2+2}}^3(z).$$

Each of the above operators is a singlet with respect to the  $\mathcal{U}_q D(2|1; \alpha)$  quantum group.

The monodromy properties of these  $\mathcal{WD}_{2|1}(k_1, k_2)$  vertex operators are therefore governed by the lower-dimensional  $\mathcal{U}_q \mathfrak{sl}(2)$  representations. For convenience, we give our quantum group conventions

and summarize the construction of elements with trivial monodromies, see [13]. The  $sl(2)$  quantum group relations are

$$\begin{aligned} K K^{-1} &= K^{-1} K = 1, \\ K E K^{-1} &= q^2 E, \quad K F K^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

with the comultiplication given by

$$(4.9) \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$(4.10) \quad \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}.$$

For a positive integer  $n$ , let  $\mathbf{V}_n$  be the  $\mathcal{U}_q sl(2)$  module with the highest-weight vector  $v_0$  such that  $E v_0 = 0$ ,  $K v_0 = q^n v_0$ , and  $F v_{i-1} = [i] v_i$  (with the standard notation  $[i] = \frac{q^i - q^{-i}}{q - q^{-1}}$ ). It follows that  $E v_i = [n - i + 1] v_{i-1}$  and  $K v_i = q^{n-2i} v_i$ .

The idea of constructing elements with trivial monodromies is to combine representations of  $\mathcal{U}_q sl(2)$  and  $\mathcal{U}_{q^{-1}} sl(2)$  quantum groups. As the basic example, we consider the  $\mathbf{V}_1$  module. Let  $\mathbf{V}'_1$  be a similar module over  $\mathcal{U}_{q^{-1}} sl(2)$ , with the basis  $v'_0$  and  $v'_1$ . *It is also a module over  $\mathcal{U}_q sl(2)$* , with the  $\mathcal{U}_q sl(2)$  action given by

$$(4.11) \quad E v'_0 = v'_1, \quad K v'_0 = q^{-1} v'_0, \quad F v'_0 = 0,$$

$$(4.12) \quad E v'_1 = 0, \quad K v'_1 = q v'_1, \quad F v'_1 = v'_0.$$

The tensor product of  $\mathcal{U}_q sl(2)$  modules  $\mathbf{V}'_1 \otimes \mathbf{V}_1$  is decomposed as

$$(4.13) \quad \mathbf{V}'_1 \otimes \mathbf{V}_1 = \mathbb{C} \oplus \mathbf{V}_2,$$

where  $\mathbf{V}_2$  is generated from  $v'_1 \otimes v_0$ , and  $\mathbf{V}_0 = \mathbb{C}$  from  $w_2 = v'_0 \otimes v_0 - q v'_1 \otimes v_1$ . This gives an element of the “ $w_{\alpha\alpha'}$ -type” in Eq. (4.4). Similar formulae can be easily written for the invariant element  $w_3 \in \mathbf{V}'_2 \otimes \mathbf{V}_2$  and  $w_{n+1} \in \mathbf{V}'_n \otimes \mathbf{V}_n$ .

**4.2. The  $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$  coset.** This coset conformal field theory can be defined as the relative semi-infinite cohomology  $H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2))$  of the complex

$$(4.14) \quad \widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} \oplus \widehat{sl}(2)_{k_3} \oplus \text{ghosts} \quad \text{with} \quad k_1 + k_2 + k_3 = -4,$$

where the ghosts are given by three free-fermion (“BC”) systems and the differential is constructed in the standard way for the diagonally embedded level-(-4)  $\widehat{sl}(2)$  algebra [14, 15] (see also [16] and references therein) starting with the differential

$$(4.15) \quad d = \oint (J^+ C_+ + J^- C_- + J^0 C_0 - 2B^0 C_+ C_- - B^- C_- C_0 + B^+ C_+ C_0)$$

that computes the absolute cohomology. The differential can be defined to act on the tensor product of the ghost modules and the vacuum representations of the three  $\widehat{sl}(2)$  algebras. For generic  $k_1$  and  $k_2$ , the cohomology of this complex is concentrated in the ghost number zero, and the vertex operator

algebra in the cohomology is the conformal field theory associated with the coset. This gives *local fields* (vacuum descendants)—the elements of  $H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \text{Vac}_{(k_1)} \otimes \text{Vac}_{(k_2)} \otimes \text{Vac}_{(k_3)})$ .

Taking cohomology elements with more general coefficients gives vertex operators of this coset theory rather than just the local fields. An obvious class  $\mathcal{C}_0$  of the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$  vertex operators are constructed as follows. One starts with vertex operators for each  $\widehat{sl}(2)$ , which are represented by  $\mathbb{C}^m(z) \otimes \mathbb{C}_q^m$ , where the first  $\mathbb{C}^m$  specifies the horizontal  $sl(2)$  subalgebra representation and  $\mathbb{C}_q^m$  the  $\mathcal{U}_q sl(2)$  quantum group representation. The reduction to the coset then consists in selecting those operators that commute with the above differential, which amounts to taking  $sl(2)$  invariants in  $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^\ell$ . This means selecting the triples  $(m, n, \ell)$  such that the tensor product  $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^\ell$  of  $sl(2)$  representations contains an invariant with respect to the diagonal  $sl(2)$  algebra. This therefore gives the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$  vertex operators that are in a 1 : 1 correspondence with the space of invariants  $[\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^\ell]^{sl(2)}$ .

Reformulating the above construction in terms of the semi-infinite cohomology immediately suggests a more general construction that gives a larger class of coset vertex operators involving descendants. Descendants of the  $(\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^\ell)(z)$  vertex operators span the tensor product of the corresponding Weyl modules,

$$(\mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_\ell)(z),$$

where  $\mathcal{M}_i$  is the Weyl module with the  $(i + 1)$ -dimensional top-level  $sl(2)$  representation and the three tensor factors are representations of  $\widehat{sl}(2)_{k_1}$ ,  $\widehat{sl}(2)_{k_2}$ , and  $\widehat{sl}(2)_{k_3}$ , respectively. The reduction then consists in calculating the cohomology spaces

$$(4.16) \quad H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_\ell)(z)$$

for each triple of positive integers  $(m, n, \ell)$ , which again amounts to a similar problem of finding  $sl(2)$  invariants (these are more numerous once further Weyl-module levels are involved beyond the top ones).

The space  $\bigoplus_{m,n,\ell} H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_\ell)(z)$  is closed under operator products and represents a subalgebra of the algebra of the  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$  vertex operators. This space, moreover, carries a representation of local fields of the coset conformal field theory because the algebra  $H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \text{Vac} \otimes \text{Vac} \otimes \text{Vac})$  acts on  $\bigoplus_{m,n,\ell} H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_\ell)$ .<sup>7</sup>

The definition of the coset as  $W = H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \text{Vac}_{(k_1)} \otimes \text{Vac}_{(k_2)} \otimes \text{Vac}_{(k_3)})$ , with its vertex operators in  $H^{\infty/2}(\widehat{sl}(2), sl(2); \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_\ell)$ , leads to another property characterizing the coset theory: the product of the coset with the  $\widehat{sl}(2)$  algebra of the level  $-k_i - 4$  that is “dual” to  $k_i$  for

<sup>7</sup>From the identification of the coset with the  $W$  algebra that we obtain below, it follows that the above construction describes those coset vertex operators that are singlets with respect to the  $\mathcal{U}_q D(2|1; \alpha)$  quantum group; only these follow in an obvious way from the definition of the coset, whereas the construction of vertex operators carrying nontrivial  $\mathcal{U}_q D(2|1; \alpha)$  representations is left for the future work.

any  $i = 1, 2, 3$  admits a vertex-operator extension to the product of the other two  $\widehat{sl}(2)_{k_i}$  algebras:

$$(4.17) \quad W \otimes \mathcal{U}\widehat{sl}(2)_{-k_1-4} \xrightarrow{\text{v.o.e.}} \mathcal{U}\widehat{sl}(2)_{k_2} \otimes \mathcal{U}\widehat{sl}(2)_{k_3}$$

(where  $\mathcal{U}$  denotes vacuum representations). This vertex-operator extension is achieved by combining the coset vertex operators described above and the standard  $\widehat{sl}(2)_{-4}$  vertex operators. The relevant mapping is in fact given by the isomorphism

$$(4.18) \quad \bigoplus_{n \geq 0} H^{\infty/2}(\widehat{sl}(2), sl(2); \mathcal{M}_n \otimes_{(k_2)} \text{Vac} \otimes_{(k_3)} \text{Vac}) \otimes \mathcal{M}_n \xrightarrow{\sim} \text{Vac}_{(k_2)} \otimes \text{Vac}_{(k_3)}$$

(with the two Weyl modules being those over the  $\widehat{sl}(2)_{k_1}$  and  $\widehat{sl}(2)_{-k_1-4}$  algebras), which involves the contraction “over the quantum group index” induced by taking the monodromy-free element in each term.

Property (4.17) characterizes  $W$  as the coset, because  $\widehat{sl}(2)_{-k_1-4}$  is then *diagonally* embedded in  $\widehat{sl}(2)_{k_2} \oplus \widehat{sl}(2)_{k_3}$  (we recall that  $k_1 + k_2 + k_3 = -4$ ), and therefore,

$$(4.19) \quad W \otimes \mathcal{U}\widehat{sl}(2)_{-k_1-4} \otimes \mathcal{U}\widehat{sl}(2)_{k_1} \longrightarrow \mathcal{U}\widehat{sl}(2)_{k_1} \otimes \mathcal{U}\widehat{sl}(2)_{k_2} \otimes \mathcal{U}\widehat{sl}(2)_{k_3},$$

where on the right-hand side, the  $\widehat{sl}(2)_{-4}$  algebra is embedded diagonally. In accordance with the definition, therefore, the relative semi-infinite cohomology of the right-hand side reproduces the coset, while on the left-hand side,  $H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2))$  evaluates as  $\mathbb{C}$  on  $\mathcal{U}\widehat{sl}(2)_{-k_1-4} \times \mathcal{U}\widehat{sl}(2)_{k_1}$  and thus gives  $W$ .

It is the property in Eq. (4.17) that we show for  $\mathcal{WD}_{2|1}(k_1, k_2)$  instead of  $W$ . Rewriting this with  $k_1$  replaced by  $k_3 = -k_1 - k_2 - 4$ , we construct in Sec. 4.3 the vertex-operator extension

$$(4.20) \quad \mathcal{WD}_{2|1}(k_1, k_2) \otimes \mathcal{U}sl(2)_{k_1+k_2} \xrightarrow{\text{v.o.e.}} \mathcal{U}\widehat{sl}(2)_{k_1} \otimes \mathcal{U}\widehat{sl}(2)_{k_2}.$$

As we have seen, this implies a homomorphism

$$(4.21) \quad \mathcal{WD}_{2|1}(k_1, k_2) \longrightarrow H^{\infty/2}(\widehat{sl}(2)_{-4}, sl(2); \text{Vac}_{(k_1)} \otimes \text{Vac}_{(k_2)} \otimes \text{Vac}_{(k_3)}),$$

which is in fact an isomorphism. Thus, constructing vertex-operator extension (4.20) will show that  $\mathcal{WD}_{2|1}(k_1, k_2)$  is indeed the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}/\widehat{sl}(2)_{k_1+k_2}$ .

**4.3. Reconstructing  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$  currents.** We now construct the vertex-operator extension (4.20) by combining  $\mathcal{WD}_{2|1}(k_1, k_2)$  vertex operators constructed in Sec. 3 with  $\widehat{sl}(2)_{k_1+k_2}$  vertex operators.

The operator in Eq. (3.39) that is a 3-dimensional representation of the  $\mathcal{U}_{q^{-1}}sl(2)$  quantum group with  $q = e^{\frac{2\pi i}{k_1+k_2+2}}$  can be contracted with the spin-1 vertex operator  $\Phi_1(k_1+k_2)$  for  $\widehat{sl}(2)_{k_1+k_2}$ . The latter is a 3-dimensional representation of  $\mathcal{U}_qsl(2)$ , which we express by writing  $\Phi_1(k_1+k_2)(z) = \mathbb{C}^3(z) \otimes \mathbb{C}_q^3$  (with the first  $\mathbb{C}^3$  factor representing the triplet with respect to the horizontal  $sl(2)$  subalgebra). This contraction “with respect to the quantum-group indices” amounts to taking a

quantum-group singlet  $w_3 \in \mathbb{C}_q^3 \otimes \mathbb{C}_{q^{-1}}^3$ . The result is then a local field in the sense of (4.4). We thus have

$$(4.22) \quad \underbrace{\mathbb{C}^3(z) \otimes \mathbb{C}_{\frac{2}{k_1+k_2+2}}^3}_{\Phi_1(k_1+k_2)(z)} \otimes \underbrace{\mathbb{C}_{\frac{-2}{k_1+k_2+2}}^3}_{\Upsilon_{113}(z)}(z) \ni \mathbb{C}^3(z) \otimes w_3.$$

The right-hand side is therefore a 3-dimensional representation of the  $sl(2)$  subalgebra of  $\widehat{sl}(2)_{k_1+k_2}$ . In this 3-dimensional space, we choose the basis  $J_*^+(z)$ ,  $J_*^0(z)$ , and  $J_*^-(z)$  such that  $J_*^+(z)$  corresponds to the highest-weight vector. We let  $J_{\text{diag}}^{\pm,0}(z)$  denote the  $\widehat{sl}(2)_{k_1+k_2}$  currents.

The space on the right-hand side of (4.22) is in fact the evaluation representation of  $\widehat{sl}(2)_{k_1+k_2}$ . Before the contraction, a number of cancellations of poles occur in the nine operator products in  $\mathbb{C}_{\frac{-2}{k_1+k_2+2}}^3(z) \cdot \mathbb{C}_{\frac{-2}{k_1+k_2+2}}^3(w)$  between the different components of  $\Upsilon_{113}$ . For the highest-weight elements in the quantum-group representations explicitly written in (3.40), we have

$$(4.23) \quad \vec{a}_1 \cdot \partial \vec{\varphi} e^{-\vec{r}_{23} \cdot \vec{\varphi}}(z) \vec{a}_1 \cdot \partial \vec{\varphi} e^{-\vec{r}_{23} \cdot \vec{\varphi}}(w) = (z-w)^{-\frac{2}{k_1+k_2+2}} \left( \frac{(\vec{a}_1 \cdot \vec{a}_1 - (\vec{a}_1 \cdot \vec{r}_{23})^2)}{(z-w)^2} + \dots \right),$$

which shows that the leading pole vanishes. After taking the quantum-group contraction, this implies that the currents

$$(4.24) \quad J_1^{\pm,0} = \frac{1}{k_1+k_2} (k_1 J_{\text{diag}}^{\pm,0} + J_*^{\pm,0}),$$

$$(4.25) \quad J_2^{\pm,0} = \frac{1}{k_1+k_2} (k_2 J_{\text{diag}}^{\pm,0} - J_*^{\pm,0})$$

satisfy the respective  $\widehat{sl}(2)_{k_i}$  algebras.<sup>8</sup> This shows (4.20).

**4.4. Reconstructing  $\widehat{sl}(2)_{k_i}$  vertex operators.** Next,  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$  vertex operators also follow from a vertex-operator extension. Let  $\Phi_{\frac{1}{2}}(k)(z) = \mathbb{C}^2(z) \otimes \mathbb{C}_{\frac{2}{k+2}}^2$  be the vertex operators for the spin- $\frac{1}{2}$  representation of  $\widehat{sl}(2)_k$ . Here,  $\mathbb{C}_{\frac{2}{k+2}}^2$  is the two-dimensional representation of  $\mathcal{U}_q sl(2)$  with the quantum group parameter  $q = e^{2\pi i/(k+2)}$  and the first  $\mathbb{C}^2$  factor represents the  $sl(2)$  doublet.

We now combine this vertex with vertex operators for  $\mathcal{W}D_{2|1}(k_1, k_2)$  that transform under the two-dimensional representation of  $\mathcal{U}_{q^{-1}} sl(2)$ . In the tensor product, there exists an  $\mathcal{U}_q sl(2)$  invariant element (in the notation of Sec. 4.1)

$$(4.26) \quad \mathbb{C}_q^2 \otimes \mathbb{C}_{q^{-1}}^2 \ni w_2 = v'_0 \otimes v_0 - q v'_1 \otimes v_1.$$

Applying this to  $\Phi_{\frac{1}{2}}(k_1+k_2)$  and  $\Psi_{212}$  as

$$(4.27) \quad \underbrace{\mathbb{C}^2(z) \otimes \mathbb{C}_{\frac{2}{k_1+k_2+2}}^2}_{\Phi_{\frac{1}{2}}(k_1+k_2)(z)} \otimes \underbrace{(\mathbb{C}_{\frac{2}{k_2+2}}^2 \otimes \mathbb{C}_{\frac{-2}{k_1+k_2+2}}^2)}_{\Psi_{212}(z)}(z) \ni \mathbb{C}^2(z) \otimes \mathbb{C}_{\frac{2}{k_2+2}}^2 \otimes w_2$$

---

<sup>8</sup>And therefore,  $J_*^{\pm,0}$  constructed above can be identified with the representation of  $\widehat{sl}(2)_{k_1+k_2}$  spanned by the  $k_2 J_1^{\pm,0}(z) - k_1 J_2^{\pm,0}(z)$  currents in  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}$ .

and similarly to  $\Phi_{\frac{1}{2}}(k_1 + k_2)$  and  $\Psi_{122}$ ,

$$(4.28) \quad \underbrace{\mathbb{C}^2(z) \otimes \mathbb{C}^2_{\frac{2}{k_1+k_2+2}}}_{\Phi_{\frac{1}{2}}(k_1+k_2)(z)} \otimes \underbrace{(\mathbb{C}^2_{\frac{2}{k_1+2}} \otimes \mathbb{C}^2_{\frac{-2}{k_1+k_2+2}})}_{\Psi_{122}(z)}(z) \ni \mathbb{C}^2(z) \otimes \mathbb{C}^2_{\frac{2}{k_1+2}} \otimes w_2,$$

we obtain the respective  $\mathbb{C}^2(z) \otimes \mathbb{C}^2_q$  structures that can be identified with the  $\widehat{sl}(2)_{k_2}$  and  $\widehat{sl}(2)_{k_1}$  spin- $\frac{1}{2}$  vertex operators  $\Phi_{\frac{1}{2}}(k_2)(z)$  and  $\Phi_{\frac{1}{2}}(k_1)(z)$ . In fact, monodromy properties alone do not guarantee that the operators constructed are necessarily primary. Showing that they are primary involves examining the relevant operator products. Pole cancellations occur in  $\mathbb{C}^3_{-\frac{2}{k_1+k_2+2}}(z) \cdot (\mathbb{C}^2_{\frac{2}{k_2+2}} \otimes \mathbb{C}^2_{-\frac{2}{k_1+k_2+2}})(w)$  in acting with  $\Upsilon_{113}(z)$  on  $\Psi_{212}(w)$ , and similarly for  $\Psi_{122}$ . For the highest-weight vectors of the respective quantum-group multiplets, we have

$$(4.29) \quad \vec{a}_1 \cdot \partial \vec{\varphi} e^{-\vec{r}_{23} \cdot \vec{\varphi}}(z) e^{-\frac{1}{2}(\vec{r}_{12} + \vec{r}_{23}) \cdot \vec{\varphi}}(w) = (z - w)^{-\frac{1}{k_1+k_2+2}} \left( \frac{-\frac{1}{2}\vec{a}_1 \cdot (\vec{r}_{12} + \vec{r}_{23})}{z - w} + \dots \right) \dots,$$

which shows that the leading pole vanishes in view of (2.31). The entire vertex operator algebra of each  $\widehat{sl}(2)_{k_i}$  can be obtained similarly (or simply by noting that components of the spin- $\frac{1}{2}$  operator generate the entire vertex operator algebra).

Thus, we have shown that the  $\mathcal{WD}_{2|1}(k_1, k_2)$  algebra defined as the commutant of  $\mathcal{U}_q D(2|1; \alpha)$  is the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$ . We next show that the same  $W$  algebra can be obtained by the Hamiltonian reduction of the affine Lie superalgebra  $\widehat{D}(2|1; \alpha)$ .

## 5. HAMILTONIAN REDUCTION OF $\widehat{D}(2|1; \alpha)$

In this section, we construct the Hamiltonian reduction  $\widehat{D}(2|1; \alpha) \rightarrow \widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$ .

**5.1. Remarks on Hamiltonian reduction.** For a Lie (super)algebra  $\mathfrak{g}$ , the general setting of the Hamiltonian reduction of  $\widehat{\mathfrak{g}}$  can be formulated as follows. For a chosen maximal nilpotent subalgebra  $\mathfrak{n} \subset \mathfrak{g}$ , one fixes an associative algebra  $\mathcal{A}$  and a homomorphism  $\mathcal{U}\widehat{\mathfrak{n}} \rightarrow \mathcal{A}$  of associative algebras. One then performs the reduction to  $\mathcal{U}\widehat{\mathfrak{g}} \times \mathcal{A}/\widehat{\mathfrak{n}}$ , i.e., the reduction with respect to the diagonal embedding

$$(5.1) \quad \begin{array}{c} \mathcal{U}\widehat{\mathfrak{g}} \otimes \mathcal{A} \\ \uparrow \\ \widehat{\mathfrak{n}} \end{array}$$

This amounts to introducing a “BRST” operator (of the type used in the semi-infinite cohomology) implementing the constraints that state the vanishing of the diagonally embedded  $\mathfrak{n}$ -valued currents. For example, the Hamiltonian reduction of  $\widehat{sl}(2)$  is reformulated in this way with  $\mathcal{A} = \mathbb{C}$  and the morphism given by  $J^+(z) \mapsto -1$ ; the embedding is therefore given by  $J^+(z) \mapsto (J^+(z), -1)$ . In more general cases,  $\mathcal{A}$  can be a free-field algebra. “Partial” Hamiltonian reductions (with only a part of the Chevalley generators constrained) can also be reformulated in this way by appropriately choosing the mapping  $\widehat{\mathfrak{n}} \rightarrow \mathcal{A}$ .

Unlike for bosonic affine Lie algebras, there is no preferred reduction scheme for superalgebras. For  $\mathfrak{g} = D(2|1; \alpha)$ , there are several “natural” choices of  $\mathcal{A}$  and of the homomorphism (an additional source of potentially different reductions is due to inequivalent choices of the maximal nilpotent subalgebra, see also Sec. 6.1). We consider the scheme of type (5.1) leading to the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$  defined in Sec. 2.2. This reduction scheme is asymmetric with respect to the three fermionic roots.<sup>9</sup>

**5.2. Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$ .** We now show that for generic values of the parameters, applying a scheme of type (5.1) to  $\widehat{D}(2|1; \alpha)$  gives the product of a Heisenberg algebra  $\mathcal{H}_0$  and the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$ . The Heisenberg algebra (a free field theory) is “trivial” piece in that it is guaranteed by the choice of  $\mathcal{A}$ , while the identification of the  $\mathcal{WD}(2|1; \alpha)$  algebra as the part of the cohomology commuting with  $\mathcal{H}_0$  is the main result of this section.

With the notation for the  $\widehat{D}(2|1; \alpha)$  algebra introduced in Appendix B, we now for brevity denote the fermionic currents corresponding to the Chevalley generators as

$$(5.2) \quad \psi_1 = \psi(-, +, +), \quad \psi_2 = \psi(+, -, +), \quad \psi_3 = \psi(+, +, -), \quad \psi_0 = \psi(+, +, +).$$

In (5.1), we take  $\mathcal{A}$  to be the algebra generated by a free fermion system  $\eta, \xi$  with the operator product

$$(5.3) \quad \eta(z) \xi(w) = \frac{1}{z - w}$$

with the mapping of the nilpotent subalgebra currents to  $\mathcal{A}$  given by  $\psi_1 \mapsto -\eta$ ,  $\psi_2 \mapsto -\eta$ ,  $\psi_3 \mapsto -\xi$  and accordingly,  $e^{(1)} \mapsto \frac{1}{2\alpha_1}$ ,  $e^{(2)} \mapsto \frac{1}{2\alpha_2}$ ,  $e^{(3)} \mapsto 0$ , and  $\psi_0 \mapsto 0$ . In terms of constraints, this is

$$(5.4) \quad \psi_1(z) - \eta(z) = 0, \quad \psi_2(z) - \eta(z) = 0, \quad \psi_3(z) - \xi(z) = 0,$$

which implies

$$(5.5) \quad e^{(1)}(z) + \frac{1}{2\alpha_1} = 0, \quad e^{(2)}(z) + \frac{1}{2\alpha_2} = 0, \quad e^{(3)}(z) = 0, \quad \psi_0(z) = 0.$$

The corresponding BRST operator is given by  $\mathcal{Q} = \oint \mathcal{J}$  with

$$(5.6) \quad \mathcal{J} = (\psi_1 - \eta)\gamma_1 + (\psi_2 - \eta)\gamma_2 + (\psi_3 - \xi)\gamma_3 + (e^{(1)} + \frac{1}{2\alpha_1})C_1 + (e^{(2)} + \frac{1}{2\alpha_2})C_2 + e^{(3)}C_3 \\ + \psi_0\gamma_0 - 2\alpha_1 B_1 \gamma_2 \gamma_3 - 2\alpha_2 B_2 \gamma_3 \gamma_1 - 2\alpha_3 B_3 \gamma_1 \gamma_2 - \beta_0 \gamma_1 C_1 - \beta_0 \gamma_2 C_2 - \beta_0 \gamma_3 C_3,$$

where we have introduced the ghosts—bosonic and fermionic first-order systems with the respective operator product expansions

$$(5.7) \quad \beta_i(z) \gamma_j(w) = \frac{-\delta_{ij}}{z - w}, \quad B_i(z) C_j(w) = \frac{\delta_{ij}}{z - w}.$$

---

<sup>9</sup>It may also be interesting to study the “symmetric” reduction, where  $\mathcal{A}$  is generated by three free-fermion systems  $b_1, c_1$ ,  $b_2, c_2$ , and  $b_3, c_3$  (with  $b_i(z)c_j(w) = \frac{\delta_{ij}}{z-w}$ ) and the mapping of the nilpotent subalgebra into  $\mathcal{A}$  is given by  $\psi(-, +, +) \mapsto b_1$ ,  $\psi(+, -, +) \mapsto b_2$ ,  $\psi(+, +, -) \mapsto b_3$ ,  $e^{(i)} \mapsto 0$  for  $i = 1, 2, 3$ , and  $\psi(+, +, +) \mapsto 0$ . This (otherwise natural) choice of the reduction scheme does not lead to the  $\mathcal{WD}(2|1; \alpha)$  algebra, and we do not expect any reduction that is symmetric with respect to the three fermionic roots to result in  $\mathcal{WD}(2|1; \alpha)$ .



Because the  $\mathcal{A}$  algebra involves the “auxiliary”  $\eta, \xi$  system, the cohomology of the BRST operator must contain a Heisenberg algebra. The current generating this algebra is readily found as

$$(5.8) \quad \widetilde{H} = 2h^{(3)} + 2B_3C_3 + \beta_0\gamma_0 + \beta_1\gamma_1 + \beta_2\gamma_2 - \beta_3\gamma_3 + \eta\xi.$$

Our aim is to find a  $W$  algebra in the ghost-number zero cohomology of  $\mathcal{Q}$  commuting with this Heisenberg algebra  $\mathcal{H}_0$ .

To this end, we use the standard procedure of combining the Sugawara energy-momentum tensor  $\mathcal{T}$  (see Eq. (B.13)) “improved” by derivatives of the Cartan currents with the ghost energy-momentum tensors. This gives the family of energy-momentum tensors in the cohomology of  $\mathcal{Q}$  in the ghost number zero,

$$(5.9) \quad \begin{aligned} \widetilde{T}_{(j)} = & \mathcal{T} + \partial B_1C_1 + \partial B_2C_2 + (2j-3)B_3\partial C_3 + 2(j-1)\partial B_3C_3 + \\ & + (j-1)\beta_0\partial\gamma_0 + j\partial\beta_0\gamma_0 + (j-2)\beta_1\partial\gamma_1 + (j-1)\partial\beta_1\gamma_1 + (j-2)\beta_2\partial\gamma_2 + (j-1)\partial\beta_2\gamma_2 + \\ & + (1-j)\beta_3\partial\gamma_3 + (2-j)\partial\beta_3\gamma_3 + (j-2)\eta\partial\xi + (j-1)\partial\eta\xi \\ & + \partial h^{(1)} + \partial h^{(2)} + 2(j-1)\partial h^{(3)}, \end{aligned}$$

where  $j$  is arbitrary. We next construct a unique combination that commutes with  $\widetilde{H}$  (and is independent of  $j$ ),

$$(5.10) \quad \widehat{T} = \widetilde{T}_{(j)} - \frac{\alpha_3}{2(\alpha_3+2)}\widetilde{H}\widetilde{H} - \frac{4(j-1) + (2j-3)\alpha_3}{2(\alpha_3+2)}\partial\widetilde{H}.$$

**Lemma 5.1.** *The central charge of  $\widehat{T}$  is given by*

$$(5.11) \quad \widehat{c} = \frac{3(\alpha_1-2)(\alpha_2-2)\alpha_3}{\alpha_1\alpha_2(\alpha_3+2)}.$$

*With the identifications*

$$(5.12) \quad \alpha_1 = k_1 + 2, \quad \alpha_2 = k_2 + 2, \quad \alpha_3 = -k_1 - k_2 - 4,$$

*this becomes the central charge of the coset theory, Eq. (1.4).*

Obviously,  $\alpha_1$  and  $\alpha_2$  can be transposed in the above formulas. We also recall (see Appendix B) that the  $\alpha$  parameter of  $\widehat{D}(2|1; \alpha)$  is determined as  $\alpha = -1 - \frac{\alpha_i}{\alpha_j}$ ; we can assume

$$(5.13) \quad \alpha = -1 - \frac{k_1 + 2}{k_2 + 2},$$

which can also be replaced by any expression obtained by transposing any two among the three levels  $k_1$ ,  $k_2$ , and  $k_3 = -k_1 - k_2 - 4$ . We note that relation (2.26) is a “semiclassical” (i.e.,  $k_1, k_2 \rightarrow \infty$ ) limit of one of these formulae.

Thus, the cohomology of  $\mathcal{Q}$  contains, in addition to  $\mathcal{H}_0$ , an algebra that contains the Virasoro algebra with the central charge of the coset theory  $\widehat{s\ell}(2)_{k_1} \oplus \widehat{s\ell}(2)_{k_2}/\widehat{s\ell}(2)_{k_1+k_2}$ . To show that this

extends to the entire  $WD_{2|1}(k_1, k_2)$  algebra commuting with  $\mathcal{H}_0$ , we introduce a filtration on the BRST complex such that

$$(5.14) \quad \mathcal{Q} = \mathcal{Q}^{(0)} + \mathcal{Q}^{(1)} + \mathcal{Q}^{(2)},$$

with  $\mathcal{Q}^{(i)}$  decreasing the filtration index by  $i$  and  $(\mathcal{Q}^{(0)})^2 = 0$ ,  $\mathcal{Q}^{(0)}\mathcal{Q}^{(1)} + \mathcal{Q}^{(1)}\mathcal{Q}^{(0)} = 0$ , and  $(\mathcal{Q}^{(1)})^2 + \mathcal{Q}^{(0)}\mathcal{Q}^{(2)} + \mathcal{Q}^{(2)}\mathcal{Q}^{(0)} = 0$ . The respective currents are given by

$$(5.15) \quad \mathcal{J}^{(0)} = \psi_1\gamma_1 + \psi_2\gamma_2 + \psi_3\gamma_3 + e^{(1)}C_1 + e^{(2)}C_2 + e^{(3)}C_3 + \psi_0\gamma_0 \\ - 2\alpha_1 B_1\gamma_2\gamma_3 - 2\alpha_2 B_2\gamma_3\gamma_1 - 2\alpha_3 B_3\gamma_1\gamma_2 - \beta_0\gamma_1C_1 - \beta_0\gamma_2C_2 - \beta_0\gamma_3C_3,$$

$$(5.16) \quad \mathcal{J}^{(1)} = -\gamma_1\eta - \gamma_2\eta - \gamma_3\xi,$$

$$(5.17) \quad \mathcal{J}^{(2)} = \frac{1}{2\alpha_1}C_1 + \frac{1}{2\alpha_2}C_2.$$

The second and the third terms in (5.14) can be viewed as “perturbations” of  $\mathcal{Q}^{(0)}$ , and  $\mathcal{Q}$  as a deformation of  $\mathcal{Q}^{(0)}$ . We now evaluate the spectral sequence associated with this decomposition of the BRST differential.

The cohomology of  $\mathcal{Q}^{(0)}$  in the ghost number zero is generated by  $\eta\xi$  and three more currents

$$(5.18) \quad \widehat{h}^{(1)} = h^{(1)} + B_1C_1 + \frac{1}{2}\beta_0\gamma_0 - \frac{1}{2}\beta_1\gamma_1 + \frac{1}{2}\beta_2\gamma_2 + \frac{1}{2}\beta_3\gamma_3,$$

$$(5.19) \quad \widehat{h}^{(2)} = h^{(2)} + B_2C_2 + \frac{1}{2}\beta_0\gamma_0 + \frac{1}{2}\beta_1\gamma_1 - \frac{1}{2}\beta_2\gamma_2 + \frac{1}{2}\beta_3\gamma_3,$$

$$(5.20) \quad \widehat{h}^{(3)} = h^{(3)} + B_3C_3 + \frac{1}{2}\beta_0\gamma_0 + \frac{1}{2}\beta_1\gamma_1 + \frac{1}{2}\beta_2\gamma_2 - \frac{1}{2}\beta_3\gamma_3.$$

The Heisenberg algebra generated by the four currents gives the ghost-number-zero part of the zeroth term in the spectral sequence. To find the first term, we note the relation

$$(5.21) \quad \mathcal{Q}^{(0)}(\frac{1}{2}\beta_2C_3\eta + \frac{1}{2}\beta_3C_2\eta - \alpha_1\gamma_0B_1\eta - \frac{1}{2}\psi(+, -, -)\eta) \\ - (\alpha_1\widehat{h}^{(1)} - \alpha_2\widehat{h}^{(2)} - (\alpha_3 + 2)\widehat{h}^{(3)} + \widetilde{H})\gamma_1\eta = -\partial(\gamma_1\eta).$$

On the cohomology of  $\mathcal{Q}^{(0)}$ , this becomes a homogeneous differential equation, and therefore, the first term in  $\mathcal{Q}^{(1)}$  acts on the cohomology of  $\mathcal{Q}^{(0)}$  as a vertex operator,

$$(5.22) \quad \gamma_1\eta = e^{\int(\alpha_1\widehat{h}^{(1)} - \alpha_2\widehat{h}^{(2)} - (\alpha_3 + 2)\widehat{h}^{(3)} + \widetilde{H})}.$$

For the other terms in  $\mathcal{Q}^{(1)}$ , we similarly find that on the cohomology of  $\mathcal{Q}^{(0)}$ , they are given by

$$(5.23) \quad \gamma_2\eta = e^{\int(-\alpha_1\widehat{h}^{(1)} + \alpha_2\widehat{h}^{(2)} - (\alpha_3 + 2)\widehat{h}^{(3)} + \widetilde{H})},$$

$$(5.24) \quad \gamma_3\xi = e^{\int(-\alpha_1\widehat{h}^{(1)} - \alpha_2\widehat{h}^{(2)} + (\alpha_3 + 2)\widehat{h}^{(3)} - \widetilde{H})}.$$

With the exponents in the last three formulae denoted as  $X_a$ ,  $a = 1, 2, 3$ , we have

$$(5.25) \quad X_1(z)X_2(w) = (\alpha_3 + 1)\log(z - w),$$

$$(5.26) \quad X_1(z)X_3(w) = (\alpha_2 - 1)\log(z - w),$$

$$(5.27) \quad X_2(z)X_3(w) = (\alpha_1 - 1)\log(z - w).$$

With the scalar product determined by the operator products, we then have  $\langle X_1, X_2 \rangle + \langle X_1, X_3 \rangle + \langle X_2, X_3 \rangle = -1$ . In addition,  $\langle X_a, X_a \rangle = 1$  because

$$(5.28) \quad X_a(z)X_a(w) = \log(z - w),$$

and therefore, *the three operators  $\gamma_1\eta$ ,  $\gamma_2\eta$ , and  $\gamma_3\xi$  acting on the ghost-number-zero part of the zeroth term of the spectral sequence are represented by fermionic screenings that are equivalent to those in Eqs. (2.21)–(2.23)*. The precise identification involves splitting the space of the four currents  $(\eta\xi, \partial X_1, \partial X_2, \partial X_3)$  into the one-dimensional space spanned by  $\widetilde{H}$  and the orthogonal complement; acting on the latter, the operators  $\gamma_1\eta$ ,  $\gamma_2\eta$ , and  $\gamma_3\xi$  single out the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$ .

Thus, the first term of the spectral sequence contains the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$  (which was defined as the vertex operator algebra in the commutant of the fermionic screenings in Eqs. (2.21)–(2.23)). Following [17], one shows that for generic values of the parameters, the cohomology of  $\mathcal{Q}$  is precisely this  $W$  algebra times  $\mathcal{H}_0$  (in particular,  $\mathcal{Q}^{(2)}$  is trivial on the first term of the spectral sequence). This completes the demonstration of the Hamiltonian reduction  $\widehat{D}(2|1; \alpha) \rightarrow \mathcal{WD}(2|1; \alpha)$ .

## 6. DISCUSSION AND CONCLUSIONS

We now briefly discuss interesting alternative constructions of the  $W$  algebra  $\mathcal{WD}(2|1; \alpha)$  corresponding to the coset  $\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2} / \widehat{sl}(2)_{k_1+k_2}$  and note several points for the future development.

**6.1.  $\mathcal{WD}(2|1; \alpha)$  from BFB screenings.** The algebra  $D(2|1; \alpha)$  has another maximal nilpotent subalgebra; it corresponds to the simple root system consisting of one fermionic and two bosonic roots. This gives another possibility for the Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$ . The result is a  $W$  algebra belonging to the same family  $\mathcal{WD}_{2|1}(k_1, k_2)$ , although the values of  $k_1$  and  $k_2$  are generically different from those resulting from the reduction in Sec. 5.

The fermionic root system corresponds, in a certain sense, to viewing  $D(2|1; \alpha)$  as three  $sl(2|1)$  algebras (with the fermionic simple roots) that pairwise intersect over  $sl(1|1)$ . At the same time, recalling that  $sl(2|1)$  admits the simple root system consisting of one bosonic and one fermionic root, we can view  $D(2|1; \alpha)$  as *two*  $sl(2|1)$  algebras (intersecting again over  $sl(1|1)$ ). The corresponding Chevalley generators of  $D(2|1; \alpha)$  are  $\psi$  (a fermion) and  $e_1$  and  $e_2$  (bosons) that satisfy, in addition to  $[\psi, \psi] = 0$ , the Serre relations

$$[e_1, e_2] = 0, \quad [e_1, [e_1, \psi]] = 0, \quad [e_2, [e_2, \psi]] = 0$$

(the pairs  $(e_1, \psi)$  and  $(e_2, \psi)$  are the simple root systems of the two  $sl(2|1)$  algebras).

The description of  $\mathcal{WD}(2|1; \alpha)$  as the commutant of one fermionic and two bosonic screenings can be deduced from the “deformation” picture in Sec. 3, where we now take the  $\widehat{sl}(2)_k$  algebra *in the standard (asymmetric) bosonization* described in Sec. 3.1. There then exist the fermionic screening

$\sigma_F = \oint e^f$  and the Dotsenko screening

$$(6.1) \quad \sigma_D = \oint a^{-k-2} e^{\sqrt{2(k+2)}\varphi},$$

where *the integrand is also an exponential*, with non-integral powers of  $a$  well-defined in terms of the bosonization in Eq. (3.9) (this screening does not exist in the bosonization used in Sec 3.3). The  $\sigma_D$  screening and the standard Wakimoto bosonization screening  $\sigma_W$  make up a *Virasoro pair*, i.e., the respective integrands are given by  $e^{\vec{\mu} \cdot \vec{\Phi}}$  and  $e^{\vec{\mu}' \cdot \vec{\Phi}}$  (with  $\vec{\Phi} = (\phi, \varphi, f)$ ), where  $\vec{\mu}' = \frac{-2}{\vec{\mu}^2} \vec{\mu}$ . The commutant of  $(\sigma_W, \sigma_F, \sigma_D)$  is the  $\widehat{sl}(2)$  algebra. The pair  $(\sigma_F, \sigma_D)$  corresponds to the  $sl(2|1)$  simple root system and, thus, generates the nilpotent subalgebra of  $\mathcal{U}_q sl(2|1)$ .

Next, the standard bosonization allows constructing the  $\tilde{\sigma}^+$  screening that makes a Virasoro pair with  $\sigma^+$ . It also *commutes with the commutant of the other screenings*. The three screenings

$$(6.2) \quad \sigma_D, \quad \sigma_F, \quad \tilde{\sigma}^+$$

(all of which have purely exponential integrands) generate the nilpotent subalgebra of  $\mathcal{U}_q D(2|1; \alpha)$  corresponding to the simple root system consisting of one fermionic and two bosonic roots and can therefore be used to single out the  $\mathcal{WD}(2|1; \alpha)$  algebra. We note that  $(\sigma_F, \tilde{\sigma}^+)$  corresponds to the simple root system of the second  $\mathcal{U}_q sl(2|1)$  quantum group.

It is instructive to identify the bosonic screenings corresponding to the three  $\mathcal{U}_q sl(2)$  quantum groups in this approach. These are  $\sigma^-$  (which is expressed as the integral of (3.13), a somewhat unconventional representation for a screening),  $\sigma_W$  (the standard Wakimoto screening), and  $\sigma^+$ .

Thus, with the “BFB” simple root system, we have used the symmetric bosonization to reproduce the quantum group content of the coset theory, the mutually commuting quantum groups  $\mathcal{U}_q D(2|1; \alpha)$  and  $\mathcal{U} sl(2)_{q'} \otimes \mathcal{U} sl(2)_{q''} \otimes \mathcal{U} sl(2)_{q'''}$ .

**6.2. Matter realization of  $\mathcal{WD}(2|1; \alpha)$ .**  $W$  algebras are vertex operator algebras selected by a set of screenings in a free-field theory. In this sense,  $W$  algebras are deformations of free-field theories with the help of operators that are *primary fields* in these theories. A natural generalization is to take a more general conformal field theory and deform it using some of its primary fields as screenings.

The  $\mathcal{WD}(2|1; \alpha)$  algebra can be realized in this spirit as a subalgebra in the product of two (universal enveloping of) Virasoro algebras and a Heisenberg algebra. We somewhat loosely refer to this system as “matter” (consisting of the Virasoro algebras with the respective central charges  $d_1$  and  $d_2$ ) dressed with a free field  $(\phi)$ . It can be arrived at by describing  $\mathcal{WD}(2|1; \alpha)$  as the commutant of the screenings in (6.2) and first evaluating the kernels of the two bosonic screenings, which gives two Virasoro algebras and a free scalar field. To construct  $\mathcal{WD}(2|1; \alpha)$  from  $\text{Vir}(d_1) \oplus \text{Vir}(d_2) \oplus (\phi)$ , we must then use the analogue of the fermionic screening in this theory. This requires generalizing the concept of the fermionic screenings (which so far have only been defined in free-field representations).

We first recall a similar construction of the  $W$  algebra  $\mathcal{W}sl(2|1)$  as a deformation of the product of a single Virasoro algebra and a free field  $\phi$  with the help of the  $\Phi_{12}$  operator. This vertex operator furnishes a two-dimensional representation of the quantum group  $\mathcal{U}_qsl(2)$ ; we let  $\Phi_{12}^\pm$  denote the two components. The operator can be dressed with a scalar such that  $\Phi_{12}^+ e^{\zeta\phi}$  becomes a fermionic operator in the sense of the operator product  $\Phi_{12}^+ e^{\zeta\phi}(z) \cdot \Phi_{12}^+ e^{\zeta\phi}(w) \sim (z-w)$ ; it then follows that  $\Phi_{12}^-$  also is a fermionic operator. It follows that the commutant of  $\oint \Phi_{12}^+ e^{\zeta\phi}$  and  $\oint \Phi_{12}^- e^{\zeta\phi}$  is the  $\mathcal{W}sl(2|1)$  algebra.

In this construction of  $\mathcal{W}sl(2|1)$ , the quantum group  $\mathcal{U}_qsl(2)$  associated with the Virasoro representation theory is extended by its two-dimensional representation  $\mathbb{C}^2$  (as before, the subscript indicates that this is a quantum group representation—the one on  $\Phi_{12}^+$  and  $\Phi_{12}^-$ —but we omit it for brevity in what follows). We observe that

$$(6.3) \quad \mathcal{U}_qsl(2) \oplus \mathbb{C}^2$$

is a part of the  $\mathcal{U}_qsl(2|1)$  quantum group, whose generators can be arranged as

$$(6.4) \quad \mathbb{C}^2 \oplus \begin{array}{c} \mathcal{U}_qsl(2) \\ \oplus \\ \mathbb{C} \end{array} \oplus \mathbb{C}^2$$

This extension of  $\mathcal{U}_qsl(2)$  to  $\mathcal{U}_qsl(2|1)$  is indeed known in the  $\widehat{sl}(2)$  representation theory [18].

Similarly to this construction of  $\mathcal{W}sl(2|1)$ , the  $\mathcal{W}D(2|1; \alpha)$  algebra is the subalgebra in the tensor product of (the universal enveloping algebras of) two Virasoro algebras and a Heisenberg algebra that commutes with the integral of the product of the respective  $\Phi_{12}$  fields dressed with a scalar,

$$(6.5) \quad \mathbf{S} = e^{\zeta\phi} \Phi_{12}^{(1)} \Phi_{12}^{(2)}.$$

This operator is represented by  $(\mathbb{C}^2 \otimes \mathbb{C}^2)(z)$ . The  $\zeta$  parameter is chosen such that the  $e^{\zeta\phi} \Phi_{12}^{(1)+} \Phi_{12}^{(2)+}$  component be a fermion (with the central charges expressed as  $d_i = 13 - 6(k_i + 2) - \frac{6}{k_i+2}$ ,  $i = 1, 2$ , and with  $\phi(z)\phi(w) = -\log(z-w)$ , this fixes  $\zeta^2 = \frac{1}{2}(k_1 + k_2 + 2)$ ). The commutant of  $\oint dz \mathbf{S}(z)$  on the product of the two Virasoro algebras and the free-field theory is the  $W$  algebra  $\mathcal{W}D(2|1; \alpha)$ . This gives the “matter” construction of the  $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$  coset.

At the quantum group level, the appearance of  $\mathcal{U}_qD(2|1; \alpha)$  in the above construction can be seen if its generators are arranged as

$$(6.6) \quad \mathcal{U}_qD(2|1; \alpha) \supset \mathbb{C} \oplus \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \begin{array}{c} \mathcal{U}_{q'}sl(2) \oplus \mathcal{U}_{q''}sl(2) \\ \oplus \\ \mathbb{C} \end{array} \oplus \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{C}$$

(where  $q'$  and  $q''$  are combined into the  $q$  and  $\alpha$  parameters of  $\mathcal{U}_qD(2|1; \alpha)$ , which therefore depends on two parameters; classically, the  $D(2|1; \alpha)$  fermions  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$  are broken to  $\mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^2$  and one of the  $sl(2)$  subalgebras to  $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ ). In the  $\mathcal{U}_{q'}sl(2) \oplus \mathcal{U}_{q''}sl(2) \oplus \mathbb{C}^2 \otimes \mathbb{C}^2$  part in (6.6), we now interpret  $\mathcal{U}_{q'}sl(2) \oplus \mathcal{U}_{q''}sl(2)$  as the quantum group symmetry of  $\text{Vir}(d_1) \oplus \text{Vir}(d_2)$ , and  $\mathbb{C}^2 \otimes \mathbb{C}^2$  as the  $\oint \mathbf{S}$  operator.

**6.3. Quantum group representations.** A manifestation of the quantum-group symmetry of conformal field theories is the correspondence between primary fields of vertex operator algebras and quantum group representations. The fusion of primary fields then corresponds to tensor products of quantum group representations. The existence of a class of quantum group representations (for example, infinite-dimensional ones) that are closed under direct sums and tensor products allows one to select a class of vertex operator algebra representations that are closed under operator products. For example, the Virasoro primary fields  $\Phi_{mn}$  are thus determined by the  $m$ - and  $n$ -dimensional representations of two  $\mathcal{U}_q\mathfrak{sl}(2)$  quantum groups with the respective  $q$  parameters determined by  $\alpha_+$  and  $\alpha_-$ ; these two quantum  $\mathfrak{sl}(2)$  groups commute with each other.

Primary fields of  $\mathcal{WD}_{2|1}(k_1, k_2)$  can be labeled as

$$\Psi_{\nu; n_1, n_2, n_3}$$

where  $\nu$  is a  $\mathcal{U}_q D(2|1; \alpha)$  (multi)index and  $n_i$  are labels of  $\mathcal{U}_{q_1}\mathfrak{sl}(2) \otimes \mathcal{U}_{q_2}\mathfrak{sl}(2) \otimes \mathcal{U}_{q_3}\mathfrak{sl}(2)$  with the respective quantum group parameters  $q_j = e^{\pi i \kappa_j}$ , see (4.5). For example, the vertex operators used in reconstructing  $\widehat{\mathfrak{sl}}(2)_{k_1}$  and  $\widehat{\mathfrak{sl}}(2)_{k_2}$  are  $\Psi_{212} = \Psi_{1;2,1,2}$  and  $\Psi_{122} = \Psi_{1;1,2,2}$ , as we saw in Sec. 3.2. These “lowest” operators are similar to  $\Phi_{21}$  for the Virasoro algebra, which are singlets with respect to one of the  $\mathcal{U}_q\mathfrak{sl}(2)$  quantum groups and doublets with respect to the other.

The two quantum groups  $\mathcal{U}_{q_1}\mathfrak{sl}(2) \otimes \mathcal{U}_{q_2}\mathfrak{sl}(2) \otimes \mathcal{U}_{q_3}\mathfrak{sl}(2)$  and  $\mathcal{U}_q D(2|1; \alpha)$  commute with each other. We note that while the presence of three  $\mathcal{U}_q\mathfrak{sl}(2)$  quantum groups is rather natural once the  $\mathcal{WD}(2|1; \alpha)$  algebra is interpreted as the  $\widehat{\mathfrak{sl}}(2)_{k_1} \oplus \widehat{\mathfrak{sl}}(2)_{k_2} / \widehat{\mathfrak{sl}}(2)_{k_1+k_2}$  coset, the emergence of  $\mathcal{U}_q D(2|1; \alpha)$  is unexpected (and somewhat mysterious) from the *coset* point of view.

**6.4. Resolutions and characters.** A further study of representations of the relevant quantum groups can lead to the construction of *resolutions* (and hence, character formulae) for  $\mathcal{WD}_{2|1}(k_1, k_2)$  representations. We concentrate on the vacuum representations, which are interesting because they describe the field content of the theory. For generic  $k_1$  and  $k_2$ , the character formula can be found by analyzing kernels of the screenings. Using the “deformation” picture in Sec. 3, one first establishes that the same  $W$  algebra  $\mathcal{WD}_{2|1}(k_1, k_2)$  is the commutant of one bosonic and two fermionic screenings provided these are chosen such that the momentum of the bosonic screening is not proportional to the sum of the two fermionic screening momenta. The commutant of the fermionic screenings is the  $\widehat{\mathfrak{sl}}(2)$  algebra, and hence,  $\mathcal{WD}_{2|1}(k_1, k_2)$  is the kernel of  $\sigma^+ = \oint J^+ e^{\beta \int J^0}$  on the weight-zero subalgebra of  $\mathcal{U}\widehat{\mathfrak{sl}}(2)$  (where the weight is the eigenvalue  $j$  of  $J_0^0$ , the zero mode of the Cartan current). The vacuum representation of  $\mathcal{WD}_{2|1}(k_1, k_2)$  can therefore be found as the kernel of  $\sigma^+$  in the weight-zero subspace of the vacuum  $\widehat{\mathfrak{sl}}(2)$  representation. By the deformation argument (taking  $\beta \rightarrow 0$ ) for generic  $k_1$  and  $k_2$ , the character of this space is the same as the character of the kernel of  $J_0^+ = \oint J^+(z)$  in the weight-zero subspace of the vacuum  $\widehat{\mathfrak{sl}}(2)$  representation (in other words, the space of invariants of the  $\mathfrak{sl}(2)$  action in the vacuum  $\widehat{\mathfrak{sl}}(2)$  representation).

Because the mapping via  $J_0^+$  is an epimorphism of the weight-0 subspace on the weight-1 subspace in the vacuum representation, the character of the kernel is given by the difference of the characters of the weight-0 and weight-1 subspaces,

$$(6.7) \quad \chi_{\text{vac}_{\mathcal{WD}(2|1; \alpha)}}(q) = \chi_{j=0}(q) - \chi_{j=1}(q).$$

Next,  $\chi_{j=0}(q)$  can be found by either studying the resolution associated with the fermionic screening, with the result

$$(6.8) \quad \chi_{j=0}(q) = \frac{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)}}{\prod_{i=1}^{\infty} (1 - q^i)^3},$$

or directly taking the  $z^0$ -component of the vacuum  $\widehat{sl}(2)$  character

$$(6.9) \quad \frac{1}{\prod_{i=1}^{\infty} (1 - zq^i) \prod_{i=1}^{\infty} (1 - z^{-1}q^i) \prod_{i=1}^{\infty} (1 - q^i)}$$

(which reproduces (6.8)). Similarly, the resolution gives

$$(6.10) \quad \chi_{j=1}(q) = \frac{\sum_{n=1}^{\infty} (-1)^{n+1} q^{\frac{1}{2}n(n+1)} + \sum_{n=2}^{\infty} (-1)^{n+1} q^{\frac{1}{2}n(n+1)-1}}{\prod_{i=1}^{\infty} (1 - q^i)^3},$$

which also is the  $z^1$ -component of (6.9).

The character of the vacuum representation of the  $W$  algebra is therefore given by

$$(6.11) \quad \chi_{\text{vac}_{\mathcal{WD}(2|1; \alpha)}}(q) = \frac{1 + 3 \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} + \sum_{n=2}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)-1}}{\prod_{i=1}^{\infty} (1 - q^i)^3} =$$

$$= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 8q^6 + 9q^7 + 19q^8 + 25q^9 + 45q^{10} + 61q^{11} + 105q^{12} + 144q^{13} + \dots$$

The coefficient 3 at  $q^4$  corresponds to the two dimension-4 descendants of the energy-momentum tensor and the  $F$  field found in Appendix A.

This character formula must also follow from a resolution constructed using three fermionic screenings. This situation is standard in that this resolution of  $\mathcal{WD}(2|1; \alpha)$ -modules has the same structure as the resolution of the trivial  $\mathcal{U}_q D(2|1; \alpha)$  representation via  $\mathcal{U}_q D(2|1; \alpha)$  Verma modules, which is given by

$$(6.12) \quad \dots \longrightarrow \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \longrightarrow \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \longrightarrow \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \longrightarrow \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \longrightarrow \bullet \longrightarrow \bullet$$

This shows that the number of modules at each term stabilizes at 4 (cf. the resolutions with a growing number of modules in each term [19] and resolutions of a more exotic shape [20]), but the details of the mappings realized by the fermionic screenings are left for a future work.

We note that there are a number of interesting “degenerate” cases corresponding to special values of  $k_1$  and  $k_2$  (see also [21] and references therein.). For example, a Felder-like complex arises if the

$\sigma^+$  screening can be applied repeatedly, e.g.,

$$(6.13) \quad \dots \xrightarrow{\sigma^+} \bullet \xrightarrow{(\sigma^+)^N} \bullet \xrightarrow{\sigma^+} \bullet \xrightarrow{(\sigma^+)^N} \dots$$

with  $(\sigma^+)^{N+1} = 0$ . The arrows, obviously, map between subspaces with different  $J_0^0$  eigenvalues. The cohomology of this complex gives the vacuum representation of the coset for the corresponding special values of  $k_1$  and  $k_2$ . These parameter values can then allow taking  $\widehat{s\ell}(2)$  representations other than the vacuum one and constructing resolutions using the corresponding weight spaces (e.g., for positive integer  $k_1$  and rational  $k_2$ ); comparing these would result in nontrivial character identities. There also exist numerous possibilities for arranging a Felder-like complex along the directions corresponding to other screenings.

**6.5. Other remarks.** The same  $W$  algebra can be obtained by reducing  $\widehat{D}(2|1; \alpha)$  with different parameters; representation-theory implications of this fact may be worth being investigated. For a fixed  $\widehat{D}(2|1; \alpha)$  algebra, on the other hand, the  $\xi$  and  $\eta$  fields can be used in the reduction constraints (cf. Eqs. (5.4)) in three different ways, which gives different  $W$  algebras from the same family,

$$(6.14) \quad \begin{array}{ccc} & \widehat{D}(2|1; \alpha) & \\ & \swarrow \quad \searrow & \\ \mathcal{WD}_{2|1}(k_1, k_2) & \longleftrightarrow & \mathcal{WD}_{2|1}(k'_1, k'_2) \end{array}$$

where the dotted line denotes a *correspondence* between (representation theories of) the two algebras, which can be interesting to explore.

The construction of  $\mathcal{WD}(2|1; \alpha)$  in terms of two Virasoro algebras dressed with a scalar suggests that the  $\widehat{D}(2|1; \alpha)$  algebra can be realized in terms of two Virasoro algebras or their supersymmetric extensions and several more free fields, similarly to the realizations of  $\widehat{s\ell}(2)$  [22] and  $\widehat{osp}(1|2)$  and  $\widehat{s\ell}(2|1)$  [23].

Because a partial Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$  is related to the  $N = 4$  superconformal algebra [4, 5], this algebra must be related to the  $\widehat{s\ell}(2) \oplus \widehat{s\ell}(2)/\widehat{s\ell}(2)$  coset via a “secondary” Hamiltonian reduction [6].

Finally, we note again that the emergence of  $\mathcal{U}_q D(2|1; \alpha)$  as a part of “quantum symmetries” of  $\mathcal{WD}_{2|1}(k_1, k_2)$  is rather mysterious when this  $W$  algebra is viewed as the coset conformal field theory. An interesting problem is to construct the  $\widehat{s\ell}(2)_{k_1} \oplus \widehat{s\ell}(2)_{k_2}/\widehat{s\ell}(2)_{k_1+k_2}$  vertex operators that carry nontrivial  $\mathcal{U}_q D(2|1; \alpha)$  representations.

**Acknowledgments.** We are grateful to A. Taormina for helpful remarks on  $N = 4$  algebras and for drawing our attention to works on the  $\widehat{D}(2|1; \alpha) \rightarrow (N=4)$  reduction. A part of the paper was written when the authors were visiting the Fields Institute, and the kind hospitality extended to us



there is gratefully acknowledged. This paper was supported in part by the RFBR Grant 01-01-00906 and the Russian Federation President Grant 99-15-96037, and also by INTAS-OPEN-97-1312 and RFBR Grant 99-01-01169.

## APPENDIX A. THE DIMENSION-FOUR FIELD

The commutant of the screenings  $\sigma_i$  constructed in Eqs. (2.2), (2.8), and (2.15)–(2.17) contains energy-momentum tensor (2.20) and possibly, higher-dimension fields generating a  $W$  algebra. We now assume  $k_1$  and  $k_2$  to be generic. For  $n = -1$  and only for  $n = -1$ , the commutant contains a primary dimension-4 field  $F$ . Using [24], this can be found as

$$\begin{aligned}
(A.1) \quad \xi^{-1} F = & \frac{K^2}{\sqrt{2k_2(k_2+2)}} \sqrt{k_1}(k_1+2)^2(k_2-1)(3k_2+4)(3k_2+11)(\partial\varphi_3)^4 \\
& - \frac{2K^{3/2}}{(k_2+2)} \sqrt{k_1}(k_1+2)^{3/2}(k_2-1)(3k_2+4)(3k_2+11)\partial^2\varphi_3(\partial\varphi_3)^2 \\
& + \frac{3\sqrt{k_1}}{2\sqrt{2k_2}}(k_1+2)(3k_2+4)(41k_2k_1^2+22k_1^2+41k_2^2k_1+186k_2k_1+88k_1+52k_2^2+168k_2+88) \times \\
& \quad \times K[(\partial\varphi_1)^2(\partial\varphi_2)^2-\partial^2\varphi_1(\partial\varphi_2)^2] \\
& + \frac{3\sqrt{k_1}}{\sqrt{2k_2}}(k_1+2)(3k_2+4)(2k_2k_1^2-11k_1^2+2k_2^2k_1-3k_2k_1-44k_1+4k_2^2-4k_2-44) \times \\
& \quad \times K[(\partial\varphi_1)^2(\partial\varphi_3)^2+(\partial\varphi_2)^2(\partial\varphi_3)^2-\partial^2\varphi_1(\partial\varphi_3)^2] \\
& + \frac{3i\sqrt{k_1}}{\sqrt{2(k_2+2)}}(k_1+2)(3k_2+4)(2k_2k_1^2-11k_1^2+2k_2^2k_1-3k_2k_1-44k_1+4k_2^2-4k_2-44)K\partial^2\varphi_2(\partial\varphi_3)^2 \\
& + \frac{\sqrt{k_1}}{2\sqrt{2(k_2+2)}}(k_1+2)(k_2-1)\sqrt{k_2}(3k_2+4)(15k_1^2+15k_2k_1+60k_1+24k_2+68)K\partial^2\varphi_3\partial^2\varphi_3 \\
& \quad - \frac{\sqrt{k_1}}{\sqrt{2}}(k_1-1)(3k_1+4)\sqrt{k_2}(5k_2^2+5k_1k_2+20k_2+4k_1+8)K\partial^3\varphi_2\partial\varphi_2 \\
& \quad - \frac{\sqrt{k_1}}{\sqrt{2(k_2+2)}}(k_1+2)(k_2-1)\sqrt{k_2}(3k_2+4)(5k_1^2+5k_2k_1+20k_1+4k_2+8)K\partial^3\varphi_3\partial\varphi_3 \\
& - \frac{\sqrt{k_1}}{k_2+2}\sqrt{k_1+2}(2k_1+k_2+4)(2k_2+3)P\sqrt{K}(\partial\varphi_1)^3\partial\varphi_3-\sqrt{k_1(k_1+2)}(3k_1+4)P\sqrt{K}\partial^2\varphi_2\partial\varphi_2\partial\varphi_3 \\
& + \frac{1}{2}\sqrt{k_1(k_1+2)}(3k_2+4)(49k_2k_1^2-22k_1^2+49k_2^2k_1+174k_2k_1-88k_1+68k_2^2+152k_2-88) \times \\
& \quad \times \sqrt{K}[\partial^2\varphi_1\partial^2\varphi_3-(\partial\varphi_1)^2\partial^2\varphi_3-(\partial\varphi_2)^2\partial^2\varphi_3] \\
& + \left(\frac{i\sqrt{k_1}}{\sqrt{k_2}}\sqrt{k_1+2}\sqrt{k_2+2}(\partial\varphi_2)^3\partial\varphi_3+\frac{3i\sqrt{k_1(k_1+2)}}{\sqrt{k_2(k_2+2)}}(k_2+1)(\partial\varphi_1)^2\partial\varphi_2\partial\varphi_3\right)(2k_1+k_2+4)P\sqrt{K} \\
& + \sqrt{k_1(k_1+2)}(3k_1+3k_2+8)P\sqrt{K}\partial^2\varphi_1\partial\varphi_1\partial\varphi_3-\frac{3i\sqrt{k_1(k_1+2)}}{\sqrt{k_2(k_2+2)}}(k_2+1)(2k_1+k_2+4)P\sqrt{K}\partial^2\varphi_1\partial\varphi_2\partial\varphi_3 \\
& - \frac{i\sqrt{k_1k_2}}{2\sqrt{k_2+2}}\sqrt{k_1+2}(3k_2+4)(49k_2k_1^2-22k_1^2+49k_2^2k_1+174k_2k_1-88k_1+68k_2^2+152k_2-88)\sqrt{K}\partial^2\varphi_2\partial^2\varphi_3 \\
& \quad - \frac{\sqrt{k_1(k_1+2)}}{2(k_2+2)}(k_2^2-k_1k_2+3k_2+4)P\sqrt{K}\partial^3\varphi_1\partial\varphi_3-\frac{i\sqrt{k_1k_2}\sqrt{k_1+2}}{2\sqrt{k_2+2}}(k_1-k_2)P\sqrt{K}\partial^3\varphi_2\partial\varphi_3 \\
& \quad + \frac{\sqrt{k_1(k_1+2)}}{12(k_2+2)}(k_2-1)k_2(3k_2+4)(5k_1^2+5k_2k_1+20k_1+4k_2+8)\sqrt{K}\partial^4\varphi_3 \\
& - \frac{\sqrt{k_1k_2}}{4\sqrt{2(k_2+2)}}(297k_2^2k_1^4+696k_2k_1^4+384k_1^4+594k_2^3k_1^3+3768k_2^2k_1^3+6336k_2k_1^3+3072k_1^3+297k_2^4k_1^2 \\
& \quad + 3768k_2^3k_1^2+14256k_2^2k_1^2+20036k_2k_1^2+9280k_1^2+696k_2^4k_1+6336k_2^3k_1+20036k_2^2k_1 \\
& \quad + 26448k_2k_1+12544k_1+384k_2^4+3072k_2^3+9280k_2^2+12544k_2+6400)(\partial\varphi_1)^4
\end{aligned}$$

$$\begin{aligned}
& + \frac{i\sqrt{2k_1}}{\sqrt{k_2+2}}(k_2+1)(3k_1^2+3k_2k_1+12k_1+5k_2+12)P(\partial\varphi_1)^3\partial\varphi_2 \\
& + \frac{i\sqrt{k_1(k_1+2)}}{2\sqrt{2(k_2+2)}}(147k_1k_2^4+204k_2^4+294k_1^2k_2^3+1142k_1k_2^3+1048k_2^3+147k_1^3k_2^2+1142k_1^2k_2^2+2300k_1k_2^2 \\
& + 1448k_2^2+204k_1^3k_2+900k_1^2k_2+856k_1k_2-16k_2-176k_1^2-704k_1-704)[(\partial\varphi_1)^2\partial^2\varphi_2-\partial^2\varphi_1\partial^2\varphi_2] \\
& + \frac{\sqrt{k_1}}{4\sqrt{2k_2}}(k_2+2)(147k_2k_1^4+132k_1^4+294k_2^2k_1^3+1440k_2k_1^3+1056k_1^3+147k_2^3k_1^2+1880k_2^2k_1^2+4872k_2k_1^2 \\
& + 3168k_1^2+572k_2^3k_1+3752k_2^2k_1+6912k_2k_1+4224k_1+496k_2^3+2336k_2^2+3504k_2+2112)(\partial\varphi_2)^4 \\
& + \frac{\sqrt{k_1k_2}}{2\sqrt{2(k_2+2)}}(75k_2^2k_1^4+210k_2k_1^4+120k_1^4+150k_2^3k_1^3+1316k_2^2k_1^3+2716k_2k_1^3+1488k_1^3+75k_2^4k_1^2 \\
& + 1316k_2^3k_1^2+6504k_2^2k_1^2+11220k_2k_1^2+6112k_1^2+210k_2^4k_1+2716k_2^3k_1+11220k_2^2k_1 \\
& + 18352k_2k_1+10432k_1+120k_2^4+1488k_2^3+6112k_2^2+10432k_2+6400)\partial^2\varphi_1(\partial\varphi_1)^2 \\
& - \frac{i\sqrt{2k_1}}{\sqrt{k_2+2}}(k_1+2)(k_2+1)(3k_1+3k_2+8)P\partial^2\varphi_1\partial\varphi_1\partial\varphi_2 \\
& + \frac{\sqrt{k_1k_2}}{4\sqrt{2(k_2+2)}}(237k_2^2k_1^4+528k_2k_1^4+288k_1^4+474k_2^3k_1^3+2804k_2^2k_1^3+4328k_2k_1^3+1952k_1^3+237k_2^4k_1^2 \\
& + 2804k_2^3k_1^2+9632k_2^2k_1^2+11860k_2k_1^2+4672k_1^2+528k_2^4k_1+4328k_2^3k_1+11860k_2^2k_1 \\
& + 12752k_2k_1+4480k_1+288k_2^4+1952k_2^3+4672k_2^2+4480k_2+1280)\partial^2\varphi_1\partial^2\varphi_1 \\
& + \frac{i\sqrt{k_1(k_2+2)}}{2\sqrt{2}}(147k_2k_1^4+132k_1^4+294k_2^2k_1^3+1292k_2k_1^3+880k_1^3+147k_2^3k_1^2+1658k_2^2k_1^2+3720k_2k_1^2+2112k_1^2 \\
& + 498k_2^3k_1+2896k_2^2k_1+4320k_2k_1+2112k_1+408k_2^3+1632k_2^2+1744k_2+704)\partial^2\varphi_2(\partial\varphi_2)^2 \\
& - \frac{\sqrt{k_1k_2}}{4\sqrt{2}}(57k_2k_1^4+60k_1^4+114k_2^2k_1^3+428k_2k_1^3+304k_1^3+57k_2^3k_1^2+836k_2^2k_1^2+1324k_2k_1^2+416k_1^2 \\
& + 468k_2^3k_1+2396k_2^2k_1+2256k_2k_1-64k_1+528k_2^3+2000k_2^2+1488k_2-320)\partial^2\varphi_2\partial^2\varphi_2 \\
& - \frac{\sqrt{k_1k_2}}{\sqrt{2}}(k_1+2)(k_1+k_2+5)(3k_1+3k_2+8)(5k_2k_1+4k_1+4k_2+8)\partial^3\varphi_1\partial\varphi_1 \\
& + \frac{i\sqrt{k_1}}{\sqrt{2(k_2+2)}}(k_2+1)(2k_1+k_2+4)P\partial^3\varphi_1\partial\varphi_2 \\
& + \frac{\sqrt{k_1k_2}}{12\sqrt{2(k_2+2)}}(15k_2^2k_1^4+42k_2k_1^4+24k_1^4+30k_2^3k_1^3+130k_2^2k_1^3+296k_2k_1^3+192k_1^3+15k_2^4k_1^2 \\
& + 130k_2^3k_1^2+432k_2^2k_1^2+1044k_2k_1^2+800k_1^2+42k_2^4k_1+296k_2^3k_1+1044k_2^2k_1 \\
& + 2192k_2k_1+1664k_1+24k_2^4+192k_2^3+800k_2^2+1664k_2+1280)\partial^4\varphi_1 \\
& - \frac{i\sqrt{k_1k_2}}{12\sqrt{2(k_2+2)}}(15k_2k_1^4+12k_1^4+30k_2^2k_1^3+70k_2k_1^3+8k_1^3+15k_2^3k_1^2+100k_2^2k_1^2+10k_2k_1^2-128k_1^2 \\
& + 42k_2^3k_1+138k_2^2k_1-88k_2k_1-224k_1+24k_2^3+56k_2^2-16k_2-64)\partial^4\varphi_2,
\end{aligned}$$

where  $K = k_1 + k_2 + 2$ ,

$$(A.2) \quad P = 37k_2k_1^2 + 44k_1^2 + 37k_2^2k_1 + 192k_2k_1 + 176k_1 + 44k_2^2 + 176k_2 + 176,$$

and

$$\begin{aligned}
(A.3) \quad \xi^{-1} &= 2\sqrt{\frac{5k_1k_2}{k_2+2}}\sqrt{(k_1-1)(k_2-1)}\sqrt{k_1+2}\sqrt{3k_1+4}\sqrt{3k_2+4} \times \\
&\quad \times \sqrt{k_1+k_2+2}\sqrt{k_1+k_2+5}\sqrt{3k_1+3k_2+8}\sqrt{P}.
\end{aligned}$$

The next operator product expansion in the  $W$  algebra is given by

$$(A.4) \quad F(z) F(w) = \frac{c/4}{(z-w)^8} + \frac{2T(w)}{(z-w)^6} + \frac{\partial T(w)}{(z-w)^5} + \frac{aTT(w) + b\partial^2 T(w) + \gamma F(w)}{(z-w)^4} \\ + \frac{aT\partial T(w) + \beta\partial^3 T(w) + \frac{\gamma}{2}\partial F(w)}{(z-w)^3} + \frac{G(w) + \dots}{(z-w)^2} + \dots$$

where  $G$  is a new *primary* dimension-6 operator (actually, one of the two dimension-6 primary operators) that looks somewhat more complicated than (A.1) (178 operator terms) and

$$a = \frac{42(k_1+2)(k_2+2)(k_1+k_2+2)}{P}, \quad b = \frac{3(k_2k_1^2 + 8k_1^2 + k_2^2k_1 + 12k_2k_1 + 32k_1 + 8k_2^2 + 32k_2 + 32)}{2P}, \\ \gamma = \frac{-3\sqrt{2/5}\mathcal{N}}{\sqrt{(k_1-1)(k_2-1)(k_1+2)(k_2+2)(3k_1+4)(3k_2+4)K(k_1+k_2+5)(3k_1+3k_2+8)P}}, \\ \beta = \frac{2(11k_2k_1^2 + 25k_1^2 + 11k_2^2k_1 + 69k_2k_1 + 100k_1 + 25k_2^2 + 100k_2 + 100)}{3P},$$

$$\mathcal{N} = 99k_2^2k_1^4 - 25k_2k_1^4 - 236k_1^4 + 198k_2^3k_1^3 + 742k_2^2k_1^3 - 672k_2k_1^3 - 1888k_1^3 + 99k_2^4k_1^2 \\ + 742k_2^3k_1^2 + 576k_2^2k_1^2 - 4088k_2k_1^2 - 5168k_1^2 - 25k_2^4k_1 - 672k_2^3k_1 - 4088k_2^2k_1 \\ - 8592k_2k_1 - 5568k_1 - 236k_2^4 - 1888k_2^3 - 5168k_2^2 - 5568k_2 - 1792.$$

## APPENDIX B. THE $\widehat{D}(2|1; \alpha)$ ALGEBRA

The Lie superalgebra  $D(2|1; \alpha)$ , or  $\mathfrak{osp}_\alpha(4|2)$ , of the dimension  $(9|8)$ , can be described as follows. Its bosonic subalgebra is the direct sum of three  $sl(2)$  algebras generated by  $e^{(i)}$ ,  $h^{(i)}$ , and  $f^{(i)}$ ,  $i = 1, 2, 3$ , with the nonvanishing commutators

$$(B.1) \quad [e^{(i)}, f^{(j)}] = 2\delta_{ij}h^{(i)}, \\ [h^{(i)}, e^{(j)}] = \delta_{ij}e^{(i)}, \\ [h^{(i)}, f^{(j)}] = -\delta_{ij}f^{(i)}.$$

The eight fermionic generators  $\psi(\beta, \gamma, \delta)$ , where  $\beta, \gamma, \delta = +, -$ , are elements of the tensor product  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$  of two-dimensional representations of the three  $sl(2)$  algebras; the specific conventions are such that

$$(B.2) \quad [e^{(1)}, \psi(-, \beta, \gamma)] = -\psi(+, \beta, \gamma), \quad [f^{(1)}, \psi(+, \beta, \gamma)] = -\psi(-, \beta, \gamma), \\ [h^{(1)}, \psi(+, \beta, \gamma)] = \frac{1}{2}\psi(+, \beta, \gamma), \quad [h^{(1)}, \psi(-, \beta, \gamma)] = -\frac{1}{2}\psi(-, \beta, \gamma)$$

and similarly for the other two  $sl(2)$  subalgebras (acting respectively on the second and the third arguments of  $\psi$ ). Finally, to write the commutation relations for the fermions, we introduce the “spinor” notation for  $sl(2)$ :

$$(B.3) \quad O_{++} = -2e, \quad O_{--} = 2f, \quad O_{+-} = O_{-+} = -2h.$$

We then have

$$(B.4) \quad [\psi(\beta_1, \beta_2, \beta_3), \psi(\gamma_1, \gamma_2, \gamma_3)] = \alpha_1 O_{\beta_1\gamma_1}^{(1)} \epsilon_{\beta_2\gamma_2} \epsilon_{\beta_3\gamma_3} + \alpha_2 O_{\beta_2\gamma_2}^{(2)} \epsilon_{\beta_1\gamma_1} \epsilon_{\beta_3\gamma_3} + \alpha_3 O_{\beta_3\gamma_3}^{(3)} \epsilon_{\beta_1\gamma_1} \epsilon_{\beta_2\gamma_2}$$

where

$$(B.5) \quad \alpha_1 + \alpha_2 + \alpha_3 = 0.$$

The algebra depends on  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  modulo a nonzero common factor. Together with (B.5) (which follows from the Jacobi identities), this leaves *one* independent parameter. In the standard notation, we thus obtain the  $D(2|1; \alpha)$  algebra with

$$(B.6) \quad \alpha = -1 - \frac{\alpha_2}{\alpha_3}.$$

The Cartan matrix can be written as

$$(B.7) \quad \begin{pmatrix} 0 & 1 & -1 - \alpha \\ 1 & 0 & \alpha \\ -1 - \alpha & \alpha & 0 \end{pmatrix}.$$

It is obviously a matter of convention which of the formulae  $\alpha = -1 - \frac{\alpha_i}{\alpha_j}$  for any ordered pair  $(\alpha_i, \alpha_j)$ ,  $i \neq j$ , is used to define  $\alpha$  instead of (B.6); therefore, the discrete transformations

$$(B.8) \quad \alpha \mapsto -\alpha - 1, \quad \alpha \mapsto \frac{1}{\alpha}, \quad \alpha \mapsto -\frac{\alpha}{\alpha + 1}, \quad \alpha \mapsto -\frac{\alpha + 1}{\alpha}, \quad \alpha \mapsto -\frac{1}{\alpha + 1}$$

leave the  $D(2|1; \alpha)$  algebra invariant.

The invariant form on  $D(2|1; \alpha)$  is given by (with  $\epsilon_{+-} = 1$ )

$$(B.9) \quad \begin{aligned} \langle e^{(i)}, f^{(j)} \rangle &= \frac{1}{\alpha_i} \delta_{ij}, \\ \langle h^{(i)}, h^{(j)} \rangle &= \frac{1}{2\alpha_i} \delta_{ij}, \\ \langle \psi(\beta_1, \beta_2, \beta_3), \psi(\gamma_1, \gamma_2, \gamma_3) \rangle &= -2\epsilon_{\beta_1\gamma_1}\epsilon_{\beta_2\gamma_2}\epsilon_{\beta_3\gamma_3}. \end{aligned}$$

We next construct the corresponding affine algebra: its generators are given by  $e_n^{(i)}$ ,  $h_n^{(i)}$ ,  $f_n^{(i)}$ ,  $\psi(\beta, \gamma, \delta)_n$ , where  $n \in \mathbb{Z}$ , and the central element. The invariant form depends on the overall scale of  $\alpha_i$ , and therefore, the pair (algebra, invariant form) depends on *two* parameters. In the corresponding affine algebra, one of these parameters can be interpreted as the level. Explicitly, the affine- $D(2|1; \alpha)$  commutation relations are given by

$$(B.10) \quad \begin{aligned} [e_m^{(i)}, f_n^{(j)}] &= 2\delta_{ij}h_{m+n}^{(i)} + \delta_{ij} m \delta_{m+n,0} \frac{1}{\alpha_i}, \\ [h_m^{(i)}, e_n^{(j)}] &= \delta_{ij}e_{m+n}^{(j)}, \\ [h_m^{(i)}, f_n^{(j)}] &= -\delta_{ij}f_{m+n}^{(j)}, \\ [h_m^{(i)}, h_n^{(j)}] &= \delta_{ij} m \delta_{m+n,0} \frac{1}{2\alpha_i}, \end{aligned}$$

by a straightforward “affinisation” of (B.2) and similar formulae for the other two  $\widehat{s\ell}(2)$  subalgebras, and by

$$(B.11) \quad \begin{aligned} [\psi(\beta_1, \beta_2, \beta_3)_m, \psi(\gamma_1, \gamma_2, \gamma_3)_n] &= \\ &= \alpha_1(O_{\beta_1\beta_1}^{(1)})_{m+n}\epsilon_{\beta_2\gamma_2}\epsilon_{\beta_3\gamma_3} + \alpha_2(O_{\beta_2\gamma_2}^{(2)})_{m+n}\epsilon_{\beta_1\gamma_1}\epsilon_{\beta_3\gamma_3} + \alpha_3(O_{\beta_3\gamma_3}^{(3)})_{m+n}\epsilon_{\beta_1\gamma_1}\epsilon_{\beta_2\gamma_2} \\ &\quad - 2m \delta_{m+n,0} \epsilon_{\beta_1\gamma_1}\epsilon_{\beta_2\gamma_2}\epsilon_{\beta_3\gamma_3}. \end{aligned}$$

The levels  $\kappa_i = \frac{1}{\alpha_i}$  of the three  $\widehat{sl}(2)$  subalgebras are therefore such that

$$(B.12) \quad \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \frac{1}{\kappa_3} = 0.$$

Any of these can be called, conventionally, the level of  $\widehat{D}(2|1; \alpha)$ , for example,  $\kappa = \kappa_3 = \frac{1}{\alpha_3}$ .

The Sugawara energy-momentum tensor  $\mathcal{T}$  is given by

$$(B.13) \quad \mathcal{T} = \alpha_1 e^{(1)} f^{(1)} + \alpha_2 e^{(2)} f^{(2)} + \alpha_3 e^{(3)} f^{(3)} + \alpha_1 h^{(1)} h^{(1)} + \alpha_2 h^{(2)} h^{(2)} + \alpha_3 h^{(3)} h^{(3)} \\ + \frac{1}{2} \psi(+, +, +) \psi(-, -, -) - \frac{1}{2} \psi(+, +, -) \psi(-, -, +) \\ - \frac{1}{2} \psi(+, -, +) \psi(-, +, -) - \frac{1}{2} \psi(-, +, +) \psi(+, -, -).$$

The central charge of the corresponding Virasoro algebra is equal to 1.

## REFERENCES

- [1] P. Bowcock, B.L. Feigin, A.M. Semikhatov, and A. Taormina,  $\widehat{sl}(2|1)$  and  $\widehat{D}(2|1; \alpha)$  as Vertex Operator Extensions of Dual Affine  $sl(2)$  Algebras, hep-th/9907171, Commun. Math. Phys. 214 (2000) 495–545.
- [2] V.G. Kač, *A sketch of Lie Superalgebra Theory*, Commun. Math. Phys. 53, 31–64 (1977).
- [3] J.F. Cornwell, *Group Theory in Physics*, Vol.3, Academic Press, 1989.
- [4] K. Ito and J. O. Madsen, *Hamiltonian Reduction and Classical Extended Superconformal Algebras*, Phys. Lett. B283 (1992) 223–230, hep-th/9202058.
- [5] K. Ito, J. O. Madsen, and J. L. Petersen, *Free Field Representations and Screening Operators for the  $N = 4$  Doubly Extended Superconformal Algebras*, Nucl. Phys. B398 (1993) 425, hep-th/9207009; *Free Field Representations of Extended Superconformal Algebras*, Phys. Lett. B292 (1992) 298–306, hep-th/9207010; *Extended Superconformal Algebras From Classical And Quantum Hamiltonian Reduction*, Workshop on “String Theory, Quantum Gravity and the Unification of the Fundamental Interactions,” Rome, September 21–26, 1992, hep-th/9211019.
- [6] J. O. Madsen and E. Ragoucy, *Secondary Quantum Hamiltonian Reductions*, Commun. Math. Phys. 185 (1997) 509–541, hep-th/9503042.
- [7] P. Goddard, A. Kent, and D. Olive, *Phys. Lett. B* 152 (1985) 88.
- [8] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, I, J. Amer. Math. Soc. 6, (1993), 905–947; II, J. Amer. Math. Soc. 6 (1993), 949–1011; III, J. Amer. Math. Soc. 7 (1994), 335–381; IV, J. Amer. Math. Soc. 7, (1994), 383–453.
- [9] P. Grozman and D. Leites, *Defining Relations for Lie Superalgebras with Cartan matrix*.
- [10] K. Gawędzki and A. Kupiainen,  *$G/H$  conformal field theory from gauged WZW model*, Phys. Lett. B215 (1988) 119–123, *Coset construction from functional integrals*, Nucl. Phys. B320 (1989) 625–668.
- [11] D. Karabali and H. Schnitzer, *BRST quantization of the gauged WZW action and coset conformal field theories*, Nucl. Phys. B329 (1990) 649–666.
- [12] S. Hwang and H. Rhedin, *The BRST formulation of  $G/H$  WZNW models*, Nucl. Phys. B406 (1993) 165–184.
- [13] C. Kassel, *Quantum Groups*, Springer, 1995.
- [14] B.L. Feigin, *The semi-infinite cohomology of the Virasoro and Kac–Moody Lie algebras*, Usp. Mat. Nauk 39:2 (1984) 195–196.
- [15] I. Frenkel, H. Garland, and G.J. Zuckerman, Proc. Natl. Acad. Sci. USA 83 (1986) 8446.
- [16] S. Hwang, *The semi-infinite cohomology of affine Lie algebras*, Commun. Math. Phys. 194 591–611 (1998), hep-th/9708100.
- [17] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Int. J. Math. Phys. A7, Supplement 1A (1992) 197–215.
- [18] B. Feigin and F. Malikov, *Modular functor and representation theory of  $\widehat{sl}_2$* , Cont. Math. 202 “Operads: Proceedings of Renaissance Conferences”; *Modular Functor and Representation Theory of  $\widehat{sl}_2$  at a Rational Level*, q-alg/9511011.

- [19] B.L. Feigin, A.M. Semikhatov, V.A. Sirota, and I.Yu. Tipunin, *Resolutions and Characters of Irreducible Representations of the  $N = 2$  Superconformal Algebras*, Nucl. Phys. B536 617–656 (1999).
- [20] B.L. Feigin and A.M. Semikhatov, *Free-Field Resolutions of the Unitary  $N = 2$  Super-Virasoro Representations*, hep-th/9810059.
- [21] A. Schilling, *Multinomials and Polynomial Bosonic Forms for the Branching Functions of the  $\widehat{su}(2)_M \times \widehat{su}(2)_N / \widehat{su}(2)_{M+N}$  Conformal Coset Models*,
- [22] A.M. Semikhatov, *The MFF singular vectors in topological conformal theories*, Mod. Phys. Lett. A9 (1994) 1867.
- [23] A.M. Semikhatov, *The Non-Critical  $N = 2$  String is an  $sl(2|1)$  Theory*, Nucl. Phys. B478 (1996) 209.
- [24] K. Thielemans, Int. J. Mod. Phys. C2 (1991) 787.

LANDAU INSTITUTE FOR THEORETICAL PHYSICS, RUSSIAN ACADEMY OF SCIENCES

TAMM THEORY DIVISION, LEBEDEV PHYSICS INSTITUTE, RUSSIAN ACADEMY OF SCIENCES